

# Non-Commutative Topology for Curved Quantum Causality

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## Abstract

A quantum causal topology is presented. This is modeled after a non-commutative scheme type of theory for the curved finitary spacetime sheaves of the non-abelian incidence Rota algebras that represent ‘gravitational quantum causal sets’. The finitary spacetime primitive algebra scheme structures for quantum causal sets proposed here are interpreted as the kinematics of a curved and reticular local quantum causality. Dynamics for quantum causal sets is then represented by appropriate scheme morphisms, thus it has a purely categorical description that is manifestly ‘gauge-independent’. Hence, a schematic version of the Principle of General Covariance of General Relativity is formulated for the dynamically variable quantum causal sets. We compare our non-commutative scheme-theoretic curved quantum causal topology with some recent  $C^*$ -quantale models for non-abelian generalizations of classical commutative topological spaces or locales, as well as with some relevant recent results obtained from applying sheaf and topos-theoretic ideas to quantum logic proper. Motivated by the latter, we organize our finitary spacetime primitive algebra schemes of curved quantum causal sets into a topos-like structure, coined ‘quantum topos’, and argue that it is a sound model of a structure that Selesnick has anticipated to underlie Finkelstein’s reticular and curved quantum causal net. At the end we conjecture that the fundamental quantum time-asymmetry that Penrose has expected to be the main characteristic of the elusive ‘true quantum gravity’ is possibly of a kinematical or structural rather than of a dynamical character, and we also discuss the possibility of a unified description of quantum logic and quantum gravity in quantum topos-theoretic terms.

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## 1. INTRODUCTION CUM PHYSICAL MOTIVATION

It has been suggested for quite some time now that not only the geometry of spacetime should be subject to quantum dynamical fluctuations, but also its topology (Wheeler, 1964). Such a proposal may be understood as saying that not only the spacetime metric, which physically represents the gravitational potential in General Relativity (GR), should be subjected to some sort of ‘quantization’ thus become a ‘quantum observable’ at Planck scales where quantum gravitational effects are expected to be significant, but also that the spacetime topology should be regarded as such a quantum dynamical variable that can in principle be observed (Isham, 1989, Finkelstein and Hallidy, 1991, Zapatin, 1998, Breslav *et al.*, 1999). Thus, the need for a quantum theory of dynamical spacetime topology has become an integral and indispensable part of our apparently never ending quest for a cogent quantum theory of gravity.

From a mathematical perspective, while a non-commutative geometry has already been proposed and significantly developed with main aim to model a quantal version of the classical spacetime geometry—a project that is supposed to be the preliminary, albeit important, step towards arriving at a mathematically well founded quantum gravity (Connes, 1994), an analogous non-commutative topology has been rather slow in coming. This may be partly attributed to our apparent inability or inertia so far at posing the ‘proper’ physical questions about ‘quantum spacetime topology’. For instance, a seemingly reasonable question one would be tempted to ask is whether the sought after quantum spacetime topology is a somehow quantized version of the classical locally Euclidean manifold topology of the spacetime of General Relativity in the same way that, say, Canonical Quantum GR purports to quantize the metric field of the classical theory of gravity (*ie*, GR). And if so, in what sense the spacetime topology may be regarded as a measurable quantum dynamical entity (*ie*, a quantum observable proper) ? Confusion about what constitutes the ‘right’ way of approaching the problem of quantum spacetime topology may be precisely due to the unphysicality, hence ‘inappropriateness’, of the questions posed in the first place.

For example, the inappropriateness of the two questions mentioned above, which are motivated by analogy to how the metrical features of classical spacetime are usually quantized, may be *prima facie* due to the following reasons: first, the fixed continuous manifold topology of GR is the main culprit for the unphysical infinities in the form of singularities that plague it long before the need to quantize gravity becomes an issue. Second, while GR provides a classical dynamics for the spacetime metric, but it fixes the spacetime topology, it does not give us even hints, let alone a classical theoretical

paradigm, of how to dynamically vary the spacetime topology<sup>1</sup>. Third, it has been seriously proposed that the ‘true quantum gravity’ or at least a sound theory for the dynamics of spacetime at quantum scales must account for the fundamental quantum time-asymmetry (Penrose, 1987, Finkelstein, 1988, Haag, 1990)<sup>2</sup>, so that the non-dynamical character of the classical manifold topology aside, its undirected or spatial traits also point to its unphysical and rather non-fundamental nature. Fourth, it is altogether conceptually doubtful whether the epithet ‘quantum’-the ‘processual’ conception of Nature as being innately dynamical and ever-fluxing, and the noun ‘topology’-the mathematical theory for an inert, static space(time) ‘out there’-can go hand in hand after all<sup>3</sup>. The latter doubt is pronounced especially in quantum gravity research where the assumption of a fixed background spacetime manifold has manifested its pathological nature both classically, due to the singularities that infest GR, and quantum mechanically, for the weaker but still troublesome infinities that assail Quantum Field Theory (QFT)<sup>4</sup>. And fifth, from the very general classical conception of ‘topology’ as ‘the study of the global features of space’, there is already a tension in terms between the ‘quantum’, which is supposed to be an effective way of looking at things at the fine, small-scale, ‘micro-local’ level, and ‘topology’, which is more likely to be of pragmatic value at the coarse, large-scale, ‘macro-global’ level of description of the world-the aforementioned temporal and spatial semantic differences of the two terms aside.

The causal set (causet) approach to the small-scale structure and dynamics of spacetime was initially conceived more-or-less with an eye towards evading the five physical impediments to theory construction presented above (Bombelli *et al.*, 1987,

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<sup>1</sup>Equivalently posited, the dynamical spacetime metric  $g_{\mu\nu}$  varies against a ‘frozen’ smooth background spacetime manifold, so that the geometry and the topology of spacetime are two fundamentally different and independent of each other structures: the first is variable, relativistic and, in principle, ‘measurable’, while the second is constant, absolute and effectively unobservable (Einstein, 1924). Furthermore,  $g_{\mu\nu}$ , as a smooth field, depends not only on the fixed continuous topological (*ie*,  $C^0$ ) structure of spacetime, but also on its  $C^\infty$ -smooth one which is also postulated or fixed up-front in GR for the sake of differential locality (Einstein, 1924).

<sup>2</sup>Penrose (1987), for example, maintains that “*the true quantum gravity is a time-asymmetric theory*”. For more on this, see section 5.

<sup>3</sup>Thus it is perhaps better to use the term ‘quantum topology’ tongue-in-cheek from now on.

<sup>4</sup>Furthermore, when GR is treated as another QFT, like in the QGR approach to quantum gravity, the non-renormalizable infinities that plague the latter appear to be insurmountable obstacles on the way to a cogent and finite quantum gravity. This has prompted many researchers in quantum gravity to regard the problem of the quantum structure of spacetime to be a first step of utmost relevance to the problem of its quantum dynamics (*ie*, quantum gravity proper). Thus, a thorough understanding of the kinematical structure of quantum spacetime will also give us invaluable clues for its dynamics (Sorkin, 1995, Mallios and Raptis, 2000) (see below and section 5).

Sorkin, 1990*a,b*, 1995, Rideout and Sorkin, 2000). For the quest of developing a cogent theory for quantum spacetime topology in particular, perhaps the most significant feature of Causet Theory (CT) is about the third point made above, namely, its insistence on a directed, causal, hence time-like or temporal conception of topology, rather than rely on the undirected, static, space-like or spatial undertones that the usual mathematical term ‘topology’ carries with it (Sorkin, 1995)<sup>5</sup>. Interestingly enough, the causet idea was born out of considerations of finitary discretizations of the classical (*ie*,  $C^0$ ) continuous manifold topology of spacetime (Sorkin, 1991, 1995) in that the basic mathematical structures—the partially ordered sets (posets)—involved in the latter remained the same, while their physical interpretation changed fundamentally from topo- or choro-logical (spatial) to chrono-logical or causal (temporal) (Sorkin, 1995, Raptis, 2000*a*, Mallios and Raptis, 2000). The partial order, when interpreted causally as the ‘after’ relation between events rather than topologically as set-theoretic inclusion between ‘elementary spatial objects’ (point-sets), is able to account for many macroscopic attributes of Lorentzian spacetime such as its topological ( $C^0$ ) and differential ( $C^\infty$ ) structure, its dimensionality (4), its signature ( $\pm 2$ ), as well as for its othochronous spin-Lorentzian (*ie*,  $SL(2, \mathbb{C})$ ) local relativity (Bombelli *et al.*, 1987, Mallios and Raptis, 2000).

On the other hand these ‘emergent classical properties’ of spacetime from a microscopic realm consisting of fundamental causet substrata may be characterized as ‘kinematical’ (*ie*, of a static structural, non-dynamical kind) (Mallios and Raptis, 2000). It is understood that if CT, or its quantum descendant QCT<sup>6</sup> (Raptis, 2000*a*, Mallios and Raptis 2000), is supposed to be a promising candidate for a finite quantum theory of gravity, then one must be able to describe a quantum dynamics for causets and their qauset relatives. Albeit, a quantum dynamical scenario for causets, and *in extenso* for qausets, has been quite slow in coming. This may be partly attributed to our difficulty in conceiving of a way of varying a poset or equivalently the incidence Rota algebra corresponding to its associated qauset (Raptis, 2000*a*)<sup>7</sup>. Similarly, the finitary spacetime sheaves (finsheaves) of qausets model for a locally finite, causal

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<sup>5</sup>See also (Finkelstein, 1988, Raptis, 2000*a*, Mallios and Raptis, 2000). Perhaps one should replace the contradictory nomen ‘quantum (spacetime) topology’ by ‘quantum causal topology’, or ‘quantum causality’ for short, thus evoke straight-away ideas about time, chronological structure, processes of change and, ultimately, dynamics.

<sup>6</sup>That is, Quantum Causal Set (or Qauset) Theory.

<sup>7</sup>Ray Sorkin in private communication (2000). Also, in (Rideout and Sorkin, 2000) the authors attribute this persistent lack of a dynamics for causets to “*the sparseness of the fundamental mathematical structure*” in that “*when all one has to work with is a discrete set and a partial order, even the notion of what we should mean by a dynamics is not obvious*”.

and quantal version of Lorentzian gravity presented in (Mallios and Raptis, 2000) has been criticized as being ‘too kinematical’ to qualify as some kind of quantum gravity proper<sup>8</sup>. The essentially kinematical character of the model was already recognized by the authors in (Mallios and Raptis, 2000) who, on the other hand, also stressed the importance of first understanding and developing the kinematics of a physical theory before tackling the problem of how to formulate the dynamics as Sorkin had previously suggested and argued for (Sorkin, 1995).

Having commented on the kinematical character of the finsheaves of gausets model for quantum gravity, at the end of (Mallios and Raptis, 2000) the following rather suggestive analogy for finding a dynamics for gausets was mentioned: as the classical topos  $\mathbf{Sh}(X)$  of sheaves of sets over a topological spacetime manifold  $X$  can be thought as a universe of continuously variable sets (Selesnick, 1991, Lambek and Scott, 1986, Bell, 1988, Mac Lane and Moerdijk, 1992), so a (quantum ?) topos of finsheaves of gausets may be thought of as a universe of dynamically variable gausets perhaps varying due to a locally finite, causal and quantal version of Lorentzian gravity still to be discovered. For conceptual reasons and interpretations coming from modern algebraic geometry and topos theory *per se*, such a topos may be regarded as a (mathematical) universe of the variable non-abelian Rota incidence algebras that model gausets-physical interpretations (*ie*, quantum dynamical interpretations of this variability) aside. In any case, the richer algebraic structure of gausets relative to the sparse<sup>9</sup> locally finite poset one of ‘classical’ causets<sup>10</sup> may be used not only to ‘enhance’ the quantum and operationally sound interpretation of these structures and their finsheaves (Raptis, 2000*a,b*, Mallios and Raptis, 2000), but it also gives us hope, mainly inspired by deep ideas and results in purely mathematical disciplines such as algebraic geometry and category theory, that topoi of finsheaves of gausets are the ‘proper’ realms for formulating and studying the quantum dynamics of space-time<sup>11</sup>. For instance, in (Mallios and Raptis, 2000) a finitary and quantal version of the Principle of General Covariance (PGC) of GR was formulated in terms of fin-sheaf morphisms such as the connection  $\mathcal{D}$ . This, it was emphasized there, implied independence or gauge invariance of the causal-topological causet dynamics from the background parameter base spacetime  $X$  on which the corresponding finsheaves are supposed to be soldered. This sheaf-theoretic formulation of the PGC is in complete

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<sup>8</sup>Chris Isham and Lee Smolin in private communication (2000).

<sup>9</sup>See quotation from (Rideout and Sorkin, 2000) in footnote 7.

<sup>10</sup>The characterization of the causets of Sorkin *et al.* as being ‘classical’ structures is based on (Raptis, 2000*a*).

<sup>11</sup>See also (Butterfield and Isham, 2000) for some possible roles that topos theory can play in quantum gravity.

analogy to the discrete version of the same principle for the dynamics of classical causets given in (Rideout and Sorkin, 2000)<sup>12</sup>.

In this paper we will not go as far as to give an explicit dynamics for qausets in the aforementioned topos of their finsheaves. We postpone this for another couple of papers that are currently in preparation (Raptis, 2000*c,d*). Instead, we will give a scheme-theoretic description of the kinematics of the curved quantum causality presented in (Mallios and Raptis, 2000) based on some deep ideas about ring or algebra localizations in algebraic geometry. The resulting scenario may be characterized as a first approach to a non-commutative topology for spacetime at small scales that can also be regarded as a rigorous mathematical formulation of the physical conception of a ‘quantum spacetime topology’.

So, the present paper is organized as follows: in section 2 we present, at the physical level of rigour, some rudiments of sheaf and scheme theory for ring and algebra localizations. We define the central notion of ‘primitive algebra schemes’. In section 3 we recast the finsheaves of qausets presented in (Mallios and Raptis, 2000) in those basic non-commutative scheme-theoretic terms. Basically, we will hold that these finsheaves are in fact primitive algebra schemes. In section 4 we discuss the physical semantics of this scheme-theoretic model of quantum causal topology with special emphasis placed on its kinematical character which is seen to be fundamentally non-commutative and directed. We also compare our scheme-theoretic non-commutative topology representing the kinematics of the curved quantum causality presented in (Mallios and Raptis, 2000) against a recent definition of ‘quantum points’ and a similar non-commutative  $C^*$ -quantale topology between them by Mulvey and Pelletier (2000)<sup>13</sup>. We also discuss in some detail some close similarities between our primitive finitary spacetime schemes of quantum causal sets models for non-commutative curved quantum causal topology with certain results about a ‘warped quantum logic’ obtained from applying sheaf and topos-theoretic ideas to quantum logic proper (Butterfield

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<sup>12</sup>The classical sequential growth-dynamics suggested in this paper is seen to be independent of the natural number ( $\mathbb{N}$ ) labeling of the causets’ vertices, which labeling is physically interpreted as a ‘gauge of external time’. In other words, two causets with the same  $\mathbb{N}$ -order of labeling of their vertices are ‘physically indistinguishable’ (*ie*, there is no external clock/time to parametrize the causets’ dynamics for it has been ‘gauged away’). Similarly, in (Mallios and Raptis, 2000) the generator of dynamics in a curved finsheaf of qausets was taken to be the sheaf morphism  $\mathcal{D}$  corresponding to a finitary spin-Lorentzian connection. Being a sheaf morphism,  $\mathcal{D}$  was seen to be causal-topologically independent of the ‘coarse gauges’  $\mathcal{U}_n$  that one could use to localize approximately or ‘coarsely measure’ the quantum causal topology of the dynamically variable qausets.

<sup>13</sup>Albeit, without a directly causal interpretation for this topology like our schematic qausets have (Raptis, 2000*a*, Mallios and Raptis, 2000). In fact, the  $C^*$ -quantales are in a very strong sense ‘spatial’ structures (Mulvey and Pelletier, 2000).

and Isham, 1998, 1999, Butterfield *et al.*, 2000). Motivated by these similarities we organize our schemes into a topos-like structure and we argue that it should be called ‘quantum topos’ after a structure that Selesnick (1991) had anticipated to underlie Finkelstein’s reticular and curved quantum causal net. In section 5 we entertain the idea that the fundamental quantum time-asymmetry expected of the ‘true quantum gravity’ (Penrose, 1987) may be of a purely kinematical character if we assume that the kinematics of a dynamical quantum causal topology is soundly represented by the curved primitive finschemes<sup>14</sup> of gausets of section 3 and their aforementioned organization into a quantum topos in section 4. We also argue that this quantum topos structure may prove to be a unifying platform for quantum logic and quantum gravity, thus vindicate Lawvere’s (1975) deep mathematical insight that algebraic geometry is in fact geometric logic and Finkelstein’s (1969, 1979, 1996) similarly fundamental physical insight that the world’s quantum logic has its origin in the dynamics of quantum spacetime<sup>15</sup>.

## 2. SHEAVES, SCHEMES AND ALGEBRA LOCALIZATIONS

Below we present at a level of rigour suitable and sufficient for our physical elaborations in the next three sections some basic elements of sheaf and scheme theory, as well as the main idea of ring and algebra localizations for which the latter theories<sup>16</sup> were primarily developed.

First, we recall from (Raptis, 2000*b*) that a sheaf  $\mathcal{S}$  of some mathematical objects  $O$  over a topological space  $X$ , written as  $\mathcal{S}(X)$ <sup>17</sup>, may be defined as a local homeomorphism  $s$  from the base space  $X$  to the sheaf or ‘sheafified’ fiber or stalk space  $\mathcal{S} := \{S\}$ <sup>18</sup>:  $s : X \rightarrow \mathcal{S}(X)$ <sup>19</sup>. When the objects  $O$  residing in the stalks  $S$  of  $\mathcal{S}(X)$

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<sup>14</sup>‘Finitary spacetime schemes’.

<sup>15</sup>See also Selesnick (1991) for a nice discussion of the close similarities between Lawvere’s Topos Theory vision of unifying logic and geometry, and Finkelstein’s Quantum Relativity vision of unifying the basic principles of the quantum and relativity theories of the world.

<sup>16</sup>Especially scheme theory.

<sup>17</sup>Or as  $\mathcal{O}(X)$ .

<sup>18</sup>The term ‘sheafification’ pertaining to the assignment of a suitable topology to the fiber space  $\mathcal{S}$  consisting of stalks or fibers of the objects  $O$  (see below). The reader should note the use of calligraphic letters for a sheaf as a collection or bundle of stalks (*ie*,  $\mathcal{S}(X)$  or  $\mathcal{O}(X)$ ) and of non-calligraphic for (the objects dwelling in) these stalks (*ie*,  $S$  or  $O$ , respectively). This distinction will be used subsequently when we define algebra schemes as sheaf-theoretic localizations of algebraic objects.

<sup>19</sup>Since  $s$  is a local homeomorphism (*ie*, a bicontinuous bijection), its inverse  $s^{-1} = \pi$  is also a local homeomorphism (Raptis, 2000*b*). The map  $\pi : \mathcal{S}(X) \rightarrow X$  is usually called ‘the projection of

have ‘extra’ algebraic structure, as for example when one considers sheaves of groups, rings, modules or algebras, it is understood that this ‘vertical’ structure raised stalk-wise in the sheaf over the points of the base space  $X$  respects or is compatible with the latter’s ‘horizontal continuity’ (*ie*, its topology). Due to the aforementioned definition of a sheaf as a local homeomorphism  $s$ , the latter may be taken as saying that the algebraic operations in a, say, algebra sheaf  $\mathcal{A}(X)$ <sup>20</sup>, are ‘locally continuous’<sup>21</sup>. Alternatively, due to the fact that as an ‘unsheafified’ or non-topologized set the algebra sheaf is the disjoint union or direct sum of its stalks,  $\mathcal{A}(X) = \bigcup_x A_x$ , one could also say that the algebra sheaf space is so topologized, or that the stalks of the sheaf are so ‘glued together’, that the local algebraic operations in each stalk  $A_x$  respect  $X$ ’s local connectivity or topology. We may summarize this to the following motto: “locally, the sheaf space  $\mathcal{S}$  and the underlying base space  $X$  are topologically equivalent or indistinguishable regardless of the extra algebraic structure that the objects  $O$  in the stalks  $S$  of the former may carry”.

In sheaf theory an important notion is that of a section of a sheaf  $\mathcal{S}(X)$ . Let  $\mathcal{U}(\mathcal{T}) = \{U\}$  be an open cover for  $X$ ’s topology  $\mathcal{T}$ , that is to say, a collection of open sets  $U$  the countable unions of finite intersections of which ‘generate’  $X$  as a topological space<sup>22</sup>. Assign to each  $U$  in  $\mathcal{U}$ , that is to say, ‘locally’ in  $X$ ’s subtopology  $\mathcal{T}(\mathcal{U})$  generated by  $\mathcal{U}$ , a class  $\Gamma$  of continuous maps  $s_U$  from  $U$  to the sheaf space  $\mathcal{S}$ , usually written as  $\Gamma(U, \mathcal{S}) = \{s_U\}$ . The  $s_U$ s in  $\Gamma(U, \mathcal{S})$  are the local, with respect to  $X$ ’s subtopology  $\mathcal{T}(\mathcal{U})$ , sections of  $\mathcal{S}(X)$ . It is a basic result in sheaf theory that a sheaf is completely determined by its (local) sections, so that the following slogan pervades the theory: “*a sheaf is its sections*” (Mallios, 1998, Raptis, 2000b). Intuitively speaking, a sheaf  $\mathcal{S}(X)$ , as a local bicontinuous bijection between the topological base space  $X$

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the sheaf on its base space’ or ‘the localization of the sheaf in  $X$ ’, or even ‘the soldering of the sheaf on  $X$ ’ (see below).

<sup>20</sup>That is, when the objects  $O$  in the stalks  $S$  of  $\mathcal{S}(X)$  are algebras  $A$ .

<sup>21</sup>For example, the algebraic product  $A_x \otimes A_x \rightarrow A_x$  vertically along the stalk  $A_x$  of  $\mathcal{A}(X)$  ( $x \in X$ ) is a continuous operation relative to how  $x$ ’s neighbouring points in  $X$  are connected to it (*ie*, with respect to  $x$ ’s local topology or connectivity). One may equivalently say that the  $\otimes_x$  structure of the stalk  $A_x$  of  $\mathcal{A}(X)$  is continuous relative to the latter’s ‘ $\pi$  point-localization index’  $x$  in  $X$ . This is secured by the  $\pi$  localization map, since  $\pi = s^{-1}$ .

<sup>22</sup>Recall that a topology  $\mathcal{T}$  for a space  $X$ , regarded as a non-topologized point-set, is a collection of subsets  $V$  of  $X$ , the so-called ‘open sets’, such that: i) the empty set  $\emptyset$  and the space  $X$  itself belong to  $\mathcal{T}$ , ii)  $\mathcal{T}$  is closed under countable unions and finite intersections. A space  $X$  equipped with a topology  $\mathcal{T} = \{V\}$  is called a topological space, symbolized as  $\mathcal{T}(X)$ . An open cover  $\mathcal{U}$  of a topological space  $\mathcal{T}(X)$  is a collection of open subsets  $U$  of  $X$  such that every  $V$  in the latter’s topology  $\mathcal{T}$  can be written as (or generated by) countable unions of finite intersections of the sets in  $\mathcal{U}$ . One says that arbitrary unions of finite intersections of the sets  $U$  in  $\mathcal{U}$  generate a subtopology  $\mathcal{T}(\mathcal{U})$  of  $\mathcal{T}(X)$  (Sorkin, 1991, Raptis, 2000b).



and the sheaf space  $\mathcal{S}$ , is determined solely by the local ‘basic’<sup>23</sup> continuous maps  $s_U$  in  $\Gamma(U, \mathcal{S})$  for all basic open sets  $U$  in  $\mathcal{U}$  covering  $X$  as a topological space in the sense described above.

Our decision to present above the sections  $\Gamma(U, \mathcal{S})$  of  $\mathcal{S}(X)$  relative to an open cover  $\mathcal{U}$  of  $X$ ’s topology  $\mathcal{T}$ , was intended with an eye towards briefly discussing the notion of ‘localization of the sheaf’s objects  $O$  with respect to the underlying space  $X$ ’ as presented in (Raptis, 2000b). Central role in this discussion is played by the notion of germ  $[s]_x$  of a (continuous) section  $s_U$  at the point  $x$  of  $X$  ( $x \in U$ ). Again, we recall from (Raptis, 2000b) that the finest basis for the topology of  $\mathcal{S}(X)$  consists of ‘irreducible’ basic open sets of the following sort  $(x, [s]_x)$ , where the second entry of the pair corresponds to the germ of a continuous section of  $\mathcal{S}(X)$ . Germs are obtained by a direct limit process of ‘infinite localization’ or refinement of a net of sets of sections of continuous functions  $\Gamma(U \in \mathcal{U}, \mathcal{S})$  defined on a corresponding inverse system of open covers for the Euclidean manifold (*ie*,  $C^0$ ) topology of  $X$ . Briefly<sup>24</sup>, by the latter we mean that as  $x$  is the product of maximum localization or refinement of (nested by inclusion) open subsets of  $X$ <sup>25</sup>, so a germ of a continuous section of  $\mathcal{S}(X)$  is simply the maximum restriction of an element  $s_U$  of  $\Gamma(U, \mathcal{S})$  to (take its values in) the stalk  $S_x$  of  $\mathcal{S}(X)$  over  $x$  ( $x \in U \subset X$ )<sup>26</sup>. Ultimately, the germs of continuous sections of the sheaf  $\mathcal{S}(X)$  of continuous functions on  $X$ , that take values in the irreducible, finest, ‘ultra local’ point-like elements of the sheaf, namely, its stalks  $S_x$ , together with the finest elements of the base space  $X$ , namely, its points  $x$ , generate (*ie*, they constitute a basis for) the topology of  $\mathcal{S}(X)$ . It follows that if the underlying topological base space  $X$  is replaced by a topologically equivalent relational structure<sup>27</sup> the germs of

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<sup>23</sup>In the sense that the  $U$ s in  $\mathcal{U}$  constitute a basis or generating set for  $X$ ’s subtopology  $\mathcal{T}(\mathcal{U})$  as explained above.

<sup>24</sup>See (Raptis, 2000b) for a more detailed and more technical account of this ‘direct limit process’ defining germs of continuous sections of sheaves and how it compares to its dual ‘inverse limit process’ of localizing the points of  $X$  from a net or inverse system of the latter’s finitary open covers as originally presented in (Sorkin, 1991). In the next two sections we will see that this ‘categorical’ duality between the inverse and direct limit processes of localization (Raptis, 2000b) will prove to be a very fruitful fact indeed.

<sup>25</sup>Fancy way of saying that the points of  $X$  are its irreducible ‘*ur*-subsets’ obtained at the limit of infinite refinement of the ‘fatter’ or ‘coarser’ open neighborhoods  $U$  about them. In this sense, “points are the elementary carriers of  $X$ ’s topology, and the  $C^0$ -manifold topology for  $X$  is its finest one attained at the inverse limit of an inverse system (poset) of its subtopologies partially ordered by inclusion” (Sorkin, 1991, Raptis, 2000b).

<sup>26</sup>Fancy way of saying that the germ of a continuous section of a sheaf is the section evaluated at a point of the base space and taking values in the stalk of the sheaf over it. In this sense, “*the stalks of the sheaf are the carriers of its topology*”-its irreducible or finest point-like elements (Raptis, 2000b).

<sup>27</sup>Like when the finitary subtopologies  $\mathcal{T}(\mathcal{U})$  of the (bounded region of the) Euclidean continuum

the continuous sections of the sheaf  $\mathcal{S}$  over the latter<sup>28</sup> preserve the local generating relations for the topology of this relational topological base space<sup>29</sup>. This last remark will prove to be important for the physical applications in the next section of the abstract scheme theory to be presented below.

It is clear from the discussion above that the concept of a sheaf  $\mathcal{S}$  of some (algebraic) objects  $O$  over a topological space  $X$  and the associated process of localization of these objects over the points of  $X$  are notions intimately related to each other. However, we saw briefly how the algebraic structure of the objects  $O$  of a sheaf  $\mathcal{S}(X)$  does not play any essential role in defining the latter's topology in that the topology of the background base space  $X$  is 'externally prescribed' (*ie*, a fixed given<sup>30</sup>) and all that is required of the (algebraic) structure of the objects of the sheaf that are soldered on it is that it respects or preserves this 'fixed background local connectivity about  $X$ 's points'<sup>31</sup>. Indeed, it would be nice to have some kind of a sheaf of algebraic objects whose topology and local properties derive from the algebraic structure of the objects themselves without any essential commitment to or dependence on a given fixed external space 'out there'. In other words, space and its (local) properties (*ie*, 'local topology') should be ideally derived from the algebraic structure of the objects of the sheaf itself and not be given up-front, *a priori* fixed once and for all by assuming *ab initio* a background topological base space on whose points these algebras are soldered (localized). From a physical point of view, and in accord with the general algebraic-

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$X$ , themselves generated by locally finite open covers  $\mathcal{U}$  of  $X$  as described above, are effectively substituted by locally finite  $T_0$  poset topologies in (Sorkin, 1991, Raptis and Zapatrin, 2000, Raptis, 2000b).

<sup>28</sup>The so-called 'finitary spacetime sheaves of continuous functions over the locally finite poset substitutes of  $X$ ' presented in (Raptis, 2000b).

<sup>29</sup>In the case of Sorkin's finitary poset substitutes of  $X$ , these generating relations for the poset topology, the so-called 'local germs of the poset topology', correspond to the transitive reduction of the partial order about each of the poset's vertices and they are precisely the immediate arrows between the vertices of the poset in its Hasse diagram (Breslav *et al.*, 1999, Raptis and Zapatrin, 2000, Raptis, 2000a, Mallios and Raptis, 2000). In (Rideout and Sorkin, 2000) these germs of the poset topology are called 'links' and in the mathematical literature they are also known as 'covering relations of the poset' in the sense that the poset topology is generated as the transitive closure of the latter (Breslav *et al.*, 1999, Raptis and Zapatrin, 2000, Raptis, 2000a).

<sup>30</sup>In the case described above, the locally Euclidean  $C^0$ -manifold topology *a priori* fixed for  $X$ .

<sup>31</sup>For example, in (Raptis, 2000b), where only the topological features of the sheaf  $\mathcal{S}(X)$  of continuous functions over a bounded region  $X$  of a spacetime  $C^0$ -manifold were of particular interest, the algebraic structure of the stalks of the sheaf played no role whatsoever in the characterization of its topology. Thus, the topology of the  $\mathcal{S}(X)$  considered in (Raptis, 2000b) merely 'imitates' the locally Euclidean topology of the given base  $C^0$ -manifold  $X$  in that the sheaf space's topology is also locally Euclidean since, by definition,  $\mathcal{S}(X)$  is a local homeomorphism from  $X$  to  $\mathcal{S}$ .

operational philosophy of quantum theory, such an idea seems very attractive, since ‘spacetime’ as a given, fixed, inert, geometric state space ‘out there’, assumed only to serve as an inert background parameter space—an external stage that indexes the dynamical propagation and interaction of physical fields, has revealed to us its ‘meta-physical’, ‘chimerical’ nature<sup>32</sup> and has prompted many thinkers to value as physically significant only (the algebraic mechanism) of our own dynamical actions of observing ‘it’<sup>33</sup> (Finkelstein, 1996), which actions can, in turn, be conveniently organized into algebra sheaves (Mallios, 1998, Raptis, 2000*b*, Mallios and Raptis, 2000).

The discussion in the last paragraph motivates the definition of a general scheme (of algebras) as follows<sup>34</sup>: let  $A$  be an associative, but not necessarily commutative, algebra. The prime spectrum of  $A$ , denoted as  $\text{Spec } A$ , is the set consisting of all prime ideals of  $A$ <sup>35</sup>. The basic idea in general scheme theory is first to ‘appropriately’ topologize  $\text{Spec } A$ , and then localize the algebra  $A$ , as a sheaf  $\mathcal{A}$  of objects  $O$  isomorphic to  $A$ , over the ‘points’ of its prime spectrum, that is to say, over its own prime ideals. In this way, the ideas in the last paragraph about an algebra sheaf over a topological space that derives from the algebra itself and which is not ‘externally prescribed’, are realized. Indeed, the underlying base space of a scheme of algebras  $A$  is taken to be its own prime spectrum  $\text{Spec } A$  suitably topologized. Then, over every prime ideal  $\mathcal{P}$  in  $\text{Spec } A$  an isomorphic copy of  $A$  is raised, written as  $A_{\mathcal{P}}$ . Subsequently, an  $A$ -algebra sheaf  $\mathcal{A}$ <sup>36</sup> is defined, as a non-topologized or non-sheafified set of stalks, to be the disjoint union of stalks of the form  $A_{\mathcal{P}}$  over each of  $\text{Spec}(A)$ ’s points  $\mathcal{P}$ ;  $\mathcal{A} := \bigcup_{\mathcal{P} \in \text{Spec } A} A_{\mathcal{P}}$ —in complete analogy to the sheaf  $\mathcal{S}(X)$  described before. Then, by giving to  $\text{Spec } A$  a fairly ‘natural’ topology<sup>37</sup>, one defines the sheaf  $\mathcal{A}(\text{Spec } A)$  as a lo-

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<sup>32</sup>Especially in numerous attempts to unite quantum mechanics with relativistic spacetime physics, as in ‘quantum gravity’ for instance, the assumption of a fixed background geometric spacetime manifold is regarded as the main culprit for the non-renormalizable infinities that afflict the theory thus render it physically unacceptable, conceptually unsatisfactory, thus deeply problematic and fundamentally incomplete.

<sup>33</sup>That is, ‘spacetime’.

<sup>34</sup>The following material is taken mainly from (Hartshorne, 1983, Shafarevich, 1994), but with occasional alterations of terminology and symbolism, as well as relevant additions and discussion to suit our exposition here. For instance, here we are interested in schemes of (not necessarily commutative) algebras and not just of abelian rings as presented in both of these books (see below).

<sup>35</sup>Recall that an ideal  $\mathcal{P}$  in an algebra  $A$  is prime whenever  $ab \in \mathcal{P} \Rightarrow (a \in \mathcal{P}) \vee (b \in \mathcal{P})$ ; ( $a, b \in A$ ).

<sup>36</sup>An  $A$ -algebra sheaf will also be called an ‘ $A$ -sheaf’ for short.

<sup>37</sup>The epithet ‘natural’ pertaining to a topology defined by using solely the algebraic structure of  $A$  as explained above. For  $R$ -ring sheaves  $\mathcal{R}$  ( $R$ -sheaves) a natural topology  $\mathcal{T}$  for  $\text{Spec } R$  is the so-called Zariski topology (Hartshorne, 1983). Here we need not digress and analyze further this most interesting of ‘closed subset topologies’ on a ring’s prime spectrum. In section 5 we will encounter another closed subset topology, albeit a non-commutative one, associated with a non-abelian  $C^*$ -

cal homeomorphism  $\alpha : \text{Spec } A \rightarrow \mathcal{A}(\text{Spec } A)$  again in complete analogy to the  $\mathcal{S}(X)$  case above. Thus, like in the case of the sheaf  $\mathcal{S}(X)$  of continuous functions on the topological manifold spacetime  $X$ , the stalks  $A_{\mathcal{P}}$  of  $\mathcal{A}(\text{Spec } A)$  are called ‘ $A$ -algebra localizations at the points  $\mathcal{P}$  of  $\text{Spec } A$ ’. It follows that the germs  $[\alpha]_{\mathcal{P}}$  of continuous (in  $\text{Spec } A$ ’s Zariski topology for example) sections of  $\mathcal{A}(\text{Spec } A)$  take values in the latter’s stalks, so that, like in  $\mathcal{S}(X)$ , the basic open sets generating  $\mathcal{A}(\text{Spec } A)$ ’s topology are of the form  $(\mathcal{P}, [\alpha]_{\mathcal{P}})$  and  $\mathcal{A}(\text{Spec } A)$  is generated by its germs of continuous sections at the points  $\mathcal{P}$  of its topological base space  $\text{Spec } A$ . Thus, like  $\mathcal{S}(X)$ , “ $\mathcal{A}(\text{Spec } A)$  is its sections” (Mallios, 1998, Raptis, 2000b).

Now we are in a position to give three formal definitions, more-or-less taken from (Hartshorne, 1983, Shafarevich, 1994), in order to arrive at the abstract, but important for our study here and the physical applications to come, notion of a ‘primitive  $A$ -scheme’ which will be amply used in the next section:

**Definition.** An  $A$ -algebraized space is a pair  $(X, \mathcal{A})$  consisting of a topological space  $X$  and a sheaf  $\mathcal{A}$  of algebras  $A$  over it. The sheaf, denoted by  $\mathcal{A}(X)$ , is called ‘the structure sheaf of  $X$ ’<sup>38</sup>.

A ‘primitive  $A$ -spectrum’ defined next is a particular instance of an  $A$ -algebraized space when the topological base space  $X$  of the structure sheaf  $\mathcal{A}(X)$  is identified with  $A$ ’s own primitive spectrum  $\mathbf{Spec } A$  provided the latter is suitably topologized<sup>39</sup>:

**Definition.** Let  $A$  be an algebra<sup>40</sup>. A primitive  $A$ -spectrum is the pair  $(\mathbf{Spec } A, \mathcal{A})$  consisting of the primitive spectrum  $\mathbf{Spec } A$  of an algebra  $A$ , which is suitably topologized, and an  $A$ -sheaf  $\mathcal{A}$  over it.

With an eye towards applying the abstract definitions of this section to our particular physical model in the next, we give the definition of a primitive  $A$ -scheme as follows:

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algebra called a ‘quantale’ (Mulvey and Pelletier, 2000).

<sup>38</sup>Subsequently, after we have defined primitive algebra schemes, we will abuse the standard terminology and instead of calling  $\mathcal{A}(X)$  ‘the structure sheaf of the base space  $X$  of the  $A$ -scheme’, we will simply call it ‘the structure sheaf of the  $A$ -scheme’.

<sup>39</sup>Recall that the primitive spectrum  $\mathbf{Spec } A$  of an algebra  $A$  is defined to be the collection of primitive ideals  $\mathcal{I}$  in  $A$ . Also recall that an ideal  $\mathcal{I}$  of an algebra  $A$  is called primitive if the factor algebra  $A/\mathcal{I}$  is simple which, in turn, means that it has no subideals other than 0 and itself. The primitive ideals of an algebra  $A$  are in 1-1 correspondence with the kernels of (equivalence classes of) its irreducible representations (irreps) (Zapatrin, 1998, Breslav *et al.*, 1999, Raptis and Zapatrin, 2000).

<sup>40</sup>Associative, but not necessarily commutative. In fact, in the following two sections the non-abelian case will interest us.

**Definition.** A primitive  $A$ -scheme is an  $A$ -algebraized space which locally looks like a primitive  $A$ -spectrum<sup>41</sup>.

The adverb ‘locally’ in the definition of a primitive scheme above requires some further explanation. Again, we follow the general lines of (Hartshorne, 1983, Shafarevich, 1994), keeping the differences in nomenclature mentioned in the last footnote in mind, and call an  $A$ -algebraized space a ‘locally  $A$ -algebraized space’ if for each point  $x$  of the base topological space  $X$  the stalk  $\mathcal{A}_x$  is an isomorphic copy of  $A$ <sup>42</sup>. Thus, the definition of a primitive  $A$ -scheme may be re-expressed as follows:

**Definition.** A primitive  $A$ -scheme is a locally primitive  $A$ -spectrum.

The last definition essentially means that for every point (*ie*, primitive ideal)  $\mathcal{I}$  in  $\mathbf{Spec} A$  there is an open neighborhood  $U(\mathcal{I}) \subset \mathbf{Spec} A$  about it such that the ‘restriction subsheaf’<sup>43</sup>  $\mathcal{A}(\mathbf{Spec} A)|_{U(\mathcal{I})}$  of  $\mathcal{A}(\mathbf{Spec} A)$  is isomorphic to a primitive  $A$ -spectrum<sup>44</sup>.

To complete the definition of primitive  $A$ -schemes above we need to explain a bit more the word ‘isomorphic’ in the last sentence. For this we first have to define the notion of  $A$ -scheme morphisms again along the general lines of Hartshorne (1983) and Shafarevich (1994):

**Definition.** A morphism of  $A$ -algebraized spaces  $(X, \mathcal{A}^1(X))$  and  $(Y, \mathcal{A}^2(Y))$  is a pair  $(f, f^\#)$  of a continuous map  $f : X \rightarrow Y$  between the underlying topological base spaces, and a map  $f^\# : \mathcal{A}^1(X) \rightarrow \mathcal{A}^2(Y)$  which is a sheaf morphism<sup>45</sup>.

**Definition.** A morphism of locally  $A$ -algebraized spaces is a morphism  $(f, f^\#)$  of  $A$ -algebraized spaces such that for each point  $x \in X$ , the induced map of local algebras (stalks)  $f_x^\# : \mathcal{A}^2(Y)|_{y=f(x)} \rightarrow \mathcal{A}^1(X)|_x$  in the respective structure sheaves is a local

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<sup>41</sup>The three definitions above may be found almost *verbatim* in either Hartshorne (1983) or Shafarevich (1994), but with rings ( $R$ ) and  $R$ -sheaves  $\mathcal{R}$  instead of algebras ( $A$ ) and  $A$ -sheaves  $\mathcal{A}$ , hence also with ‘ $R$ -ringed’ spaces instead of ‘ $A$ -algebraized’ ones, as well as with prime spectra  $\mathbf{Spec} R$  of  $R$ s instead of primitive ones  $\mathbf{Spec} A$  of  $A$ s. With these substitutions, the third definition of primitive schemes above corresponds to the standard definition of a (general) scheme in (commutative) algebraic geometry.

<sup>42</sup>The definition of a ‘locally primitive  $A$ -spectrum’, as a particular kind of a locally  $A$ -algebraized space, follows directly from this.

<sup>43</sup>For the notion of a subsheaf of a sheaf see (Hartshorne, 1983, Mallios, 1998).

<sup>44</sup>The analogue in (Hartshorne, 1983, Shafarevich, 1994) of our primitive  $A$ -spectrum is a so-called ‘affine  $R$ -scheme’, while that of our primitive  $A$ -scheme, a ‘general  $R$ -scheme’ or simply ‘ $R$ -scheme’. Thus a general scheme is a locally affine scheme.

<sup>45</sup>For the definition of sheaf morphisms see (Hartshorne, 1983, Mallios, 1998).

homomorphism of local algebras<sup>46</sup>.

A morphism  $(f, f^\#)$  between two locally  $A$ -algebraized spaces is called an ‘isomorphism’ if  $f$  is a homeomorphism of the underlying topological base spaces and  $f^\#$  an isomorphism of sheaves<sup>47</sup>.

The definitions above, when applied to the case of primitive  $A$ -schemes, define primitive  $A$ -scheme morphisms and isomorphisms, respectively<sup>48</sup>. Finally, a pair  $(f, f^\#)$  of endomorphic maps  $f : \mathbf{Spec} A \rightarrow \mathbf{Spec} A$  and  $f^\# : \mathcal{A}(\mathbf{Spec} A) \rightarrow \mathcal{A}(\mathbf{Spec} A)$ , so that  $f$  is a homeomorphism and  $f^\#$  a sheaf isomorphism, is called a ‘primitive  $A$ -scheme automorphism’. From the discussion about germs of sheaves before it follows that a primitive  $A$ -scheme automorphism preserves the germs of the local topology of the topological base space  $\mathbf{Spec} A$ , and in the structure  $A$ -sheaf  $\mathcal{A}$  over the latter, the germs of its continuous (in  $\mathbf{Spec} A$ ’s topology) local sections. In particular, when a relational topology  $\mathcal{T}$ , such as a poset topology, is given to  $\mathbf{Spec} A$ <sup>49</sup>, a primitive  $A$ -scheme automorphism preserves the latter’s germ-relations or links<sup>50</sup>, as well as the corresponding germs of  $\mathcal{A}$ ’s continuous local sections which, by the definition of  $\mathcal{A}$  as a local homeomorphism, map the germ-relations of the topological base space to similar generating relations in the structure sheaf space  $\mathcal{A}$ .

In closing this section we follow (Hartshorne, 1983) and alternatively refer to the topological base space  $X$  without its overlying scheme structure as the ‘space of  $X$ ’, and write  $sp(X)$ . It is the essence of scheme theory that for schematic  $A$ -algebra localizations,  $sp(X)$  derives from the algebra itself and it is not given ‘from outside’ as explained above. In the next section we will see how point-events and a quantum causal topology on them may be extracted by a so-called ‘Gel’fand spatialization procedure’ from the incidence algebras representing qausets which, in turn, will be seen to be localized, as primitive  $A$ -schemes, over  $\mathbf{Spec} A$ . Hence in the latter schemes,  $sp(X) \equiv \mathbf{Spec} A$ , and the space of the scheme is extracted by spatialization from the very  $A$ -stalks of its structure sheaf  $\mathcal{A}$ .

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<sup>46</sup>See (Hartshorne, 1983) for more details on this. Note in particular that if  $\mathcal{A}_x^1$  and  $\mathcal{A}_{y=f(x)}^2$  are the localizations of an algebra  $A^1$  over  $x \in X$  and of another algebra  $A^2$  over  $y \in Y$  in the respective locally  $A$ -algebraized spaces (*ie*, the stalks of the corresponding structure sheaves  $\mathcal{A}^1(X)$  and  $\mathcal{A}^2(Y)$  over  $x$  and  $y$ , respectively), the algebra homomorphism  $\phi : \mathcal{A}_x^1 \rightarrow \mathcal{A}_y^2$  between them is called ‘local’ if it preserves their maximal ideals  $\mathcal{M}$  as follows:  $\phi^{-1}(\mathcal{M}_y^2) = \mathcal{M}_x^1$  ( $\mathcal{M}_x^1 \in \text{Max } \mathcal{A}_x^1$ ,  $\mathcal{M}_y^2 \in \text{Max } \mathcal{A}_y^2$ ).

<sup>47</sup>That is,  $f^\#$  is a bijective sheaf morphism (Mallios, 1998).

<sup>48</sup>Note in particular that a primitive  $A$ -scheme morphism is simply required to carry, stalk-wise, primitive ideals from the  $\mathcal{A}_{\mathcal{T}_2}^2$  stalk to primitive ideals in the  $\mathcal{A}_{\mathcal{T}_1}^1$  stalk in such a way that the local topology of the respective primitive spectra base spaces is preserved.

<sup>49</sup>As will be given to  $\mathbf{Spec} A$  in the following section.

<sup>50</sup>See footnote 28.

As we said earlier, in the following section we present the curved finsheaves of qausets in (Mallios and Raptis, 2000) as primitive  $A$ -schemes as defined above.

### 3. NON-COMMUTATIVE PRIMITIVE $\vec{\Omega}$ -SCHEMES OF QAUSSETS

In this section we plan to give a brief history of QCT picking from each stage of its development the elements of the theory that are of immediate interest to our work here, also giving the corresponding references to the literature. This selective review of QCT will culminate in presenting the curved finsheaves of qausets in (Mallios and Raptis, 2000) as primitive  $A$ -schemes and it will highlight the way in which very general physical considerations are able to determine the precise mathematical structure of a model for (at least the kinematics of<sup>51</sup>) a physical theory-in our case ‘Quantum Causal Dynamics’ (QCD)<sup>52</sup>.

As it was mentioned in the introduction, CT (Bombelli *et al.*, 1987, Sorkin, 1990*a,b*) was originally motivated by discrete approximations of the continuous (*ie*,  $C^0$ ) topology of a spacetime manifold (Sorkin, 1995). These so-called ‘finitary substitutes’ of the  $C^0$ -topology of spacetime were seen to be posets having the structure of  $T_0$  topological spaces (Sorkin, 1991)<sup>53</sup>. The physical interpretation of these structures were as locally finite approximations of the locally Euclidean manifold topology of (a bounded region  $X$  of) classical spacetime, whereby, a spacetime point-event in the latter is effectively substituted by a ‘coarse’ open neighborhood about it and the region  $X$  is covered by a locally finite number of the latter. From such locally finite open covers  $\mathcal{U}$  of  $X$ <sup>54</sup>, Sorkin (1991) extracted by a suitable ‘algorithm’ the aforementioned ‘topological posets’. The soundness of the interpretation of topological posets as finitary replacements of the continuum  $X$  rests on the fact that an inverse system of the latter possesses an inverse limit topological space which is homeomorphic to  $X$ . This limit may alternatively be stated as follows: the  $C^0$ -topology of  $X$  is recovered at maximum localization or refinement of  $X$ ’s point-events (Raptis, 2000*b*).

Indeed, with every finitary poset substitute of  $X$  a sheaf of appropriately defined continuous functions (on  $X$ ) was defined in such a way that, as a space on its own,

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<sup>51</sup>See (Mallios and Raptis, 2000) and section 5.

<sup>52</sup>QCD is the locally finite, causal and quantal version of Lorentzian gravity (Mallios and Raptis, 2000) whose acronym should not be confused with the standard one for Quantum Chromodynamics.

<sup>53</sup>These posets may be called ‘topological posets’ (Raptis, 2000*a*) and the topological spaces that they stand for ‘relational topologies’, the relation being a partial order. See discussion earlier around footnote 27.

<sup>54</sup>An open cover  $\mathcal{U}$  of  $X$  is said to be locally finite if for each point  $x$  of  $X$  there is an open neighborhood about it that meets a finite number of the covering open sets in  $\mathcal{U}$ .

was seen to be locally homeomorphic to the poset, thus, technically speaking, a sheaf over it (Raptis, 2000b). These structures were coined ‘finitary spacetime sheaves’ (finsheaves) and their physical interpretation was as locally finite or ‘coarse’ approximations of the  $C^0$ -spacetime observables. The soundness of this interpretation rests on the fact that an inverse system of such finsheaves was seen to yield, again at the limit of maximum localization or refinement of  $X$  into its point-events, a space homeomorphic to  $\mathcal{S}(X)$ -the sheaf of continuous functions on  $X$ . Thus, there is a significant change of emphasis in the physical interpretation of finsheaves in comparison to that of finitary substitutes: from rough approximations of spacetime point-events in (Sorkin, 1991), to coarse approximations of (algebras of) operations of localization of spacetime point-events and of their locally Euclidean manifold topology, as these (algebraic) operations reside in the (algebra) stalks of the corresponding finsheaves (Raptis, 2000b)<sup>55</sup>. From the discussion in the previous section about sheaves whose base spaces are relational topologies, it follows that the germs of the finsheaves in (Raptis, 2000b) preserve the links or immediate arrows of their underlying  $T_0$  topological poset base spaces.

In (Raptis and Zapatin, 2000) an algebraic quantization procedure for the finitary relational poset topological spaces of (Sorkin, 1991) was presented. This essentially involved the association with every finitary poset substitute  $P$  of the continuous manifold  $X$  of an algebra  $\Omega(P)$ , the so-called ‘incidence algebra of the poset’ (Rota, 1968), and the ‘dual’ assignment<sup>56</sup> of a poset  $P(\Omega)$  to an arbitrary finite dimensional algebra  $\Omega$ . The first association we may formally represent by the arrow  $P \xrightarrow{\omega} \Omega(P)$ , while its dual or ‘opposite’<sup>57</sup> by the arrow  $\Omega \xrightarrow{p} P(\Omega)$ . We first present the  $p$ -correspondence,

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<sup>55</sup>The words ‘algebra’ and ‘algebraic’ are put in parentheses above because, as it was mentioned in the previous section, in (Raptis, 2000b) no allusion to the particular algebraic structure of the stalks of the sheaf  $\mathcal{S}(X)$  was made. We just note with an eye towards the next section that  $\mathcal{S}(X)$  is usually taken to be the ‘commutative sheaf’  $\mathcal{C}^0(X)$  of abelian  $C^*$ -algebras  $C^0(X, \mathbb{C})$  of continuous complex-valued functions on  $X$ , so that the locally Euclidean manifold topology of  $X$  is usually identified with that of the sheaf  $\mathcal{C}^0$ . Due to this identification we used ‘locally Euclidean manifold topology’ and ‘ $C^0$ -topology’ interchangeably above. Also due to this identification, the locally Euclidean manifold topology of  $X$  may be characterized as ‘commutative’ or ‘classical’-effectively a ‘locale’ (Mac Lane and Moerdijk, 1992, Mulvey and Pelletier, 2000, see next section) which is regarded as a generalization of classical topological spaces. It follows that a sheaf of non-commutative algebras, playing the role of the structure sheaf of a non-commutative scheme, may be associated with a ‘non-commutative’ or ‘quantal’ topological base space-a ‘quantale’ (Mulvey and Pelletier, 2000; see next section).

<sup>56</sup>The epithet ‘dual’ is given in the categorical sense of the word, that is to say, it means that the two maps corresponding to these dual to each other assignments will be seen subsequently to be contravariant functors between the respective categories of finitary posets/poset morphisms and of incidence algebras/algebra homomorphisms associated with them.

<sup>57</sup>The epithet ‘inverse’ could also be used instead of ‘opposite’. We will see later that in a subtle



then the  $\omega$  one.

The  $p$ -association  $\Omega \xrightarrow{p} P(\Omega)$  is of special interest to us here and corresponds to a construction originally presented in detail in (Zapatrin, 1998) called ‘Gel’fand spatialization procedure’<sup>58</sup>. We briefly review it below: let  $\Omega$  be a finite dimensional, associative, non-abelian algebra. Let  $p$  and  $q$  be two (equivalence classes of) irreps of  $\Omega$ <sup>59</sup> whose kernels  $p^{-1}(0)$  and  $q^{-1}(0)$  are primitive ideals  $\mathcal{I}_p$  and  $\mathcal{I}_q$  in it (*ie*,  $\mathcal{I}_p, \mathcal{I}_q \in \mathbf{Spec} \Omega$ ). One regards as vertices of  $P(\Omega)$  the points of  $\mathbf{Spec} \Omega$  and builds the immediate arrows or links between them according to the following rule

$$p \xrightarrow{*} q \Leftrightarrow \mathcal{I}_p \rho \mathcal{I}_q := \mathcal{I}_p \mathcal{I}_q (\neq \mathcal{I}_q \mathcal{I}_p) \not\subset \mathcal{I}_p \cap \mathcal{I}_q \quad (1)$$

where  $\mathcal{I}_p \mathcal{I}_q$  is understood as the product of subsets of  $\Omega$ <sup>60</sup> and  $\mathcal{I}_p \cap \mathcal{I}_q$  is their ‘intersection ideal’ in  $\Omega$ .  $\xrightarrow{*}$  represents links in  $P$ , while  $\rho$  is the generator or germ of  $\Omega$ ’s Rota topology  $\mathcal{T}_\rho(\mathbf{Spec} \Omega)$ <sup>61</sup> (Breslav *et al.*, 1999, Raptis and Zapatrin, 2000, Raptis, 2000a, Mallios and Raptis, 2000).

In fact, the identification in (1) of  $P$ ’s immediate arrows or links  $\xrightarrow{*}$  with the germ  $\rho$  of  $\mathcal{T}(\mathbf{Spec} \Omega)$  is a theorem<sup>62</sup> which can be stated thus: the Sorkin  $T_0$  poset topology of  $P(\Omega)$  is obtained as the transitive closure of  $\xrightarrow{*}$  when the latter is identified with the germ  $\rho$  of  $\Omega$ ’s Rota topology<sup>63</sup>. Thus, the Gel’fand spatialization procedure  $\Omega \xrightarrow{p} P(\Omega)$  first consists of suitably topologizing  $\mathbf{Spec} \Omega$ <sup>64</sup>, then builds a topological poset on it whose links between its points<sup>65</sup> are drawn precisely when the generating relation  $\rho$  holds between the corresponding ideals in  $\mathbf{Spec} \Omega$ . It is important to remark that when the algebra  $\Omega$  is commutative, the Rota topology is trivial in the sense that it is

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sheaf-theoretic sense the maps  $\omega$  and  $p$  are inverse of each other and are the finitary correspondents of the ‘sheafifying map’  $s : X \rightarrow \mathcal{S}(X)$  and its inverse ‘soldering map’  $\pi : \mathcal{S}(X) \rightarrow X$ , respectively, that we saw in the previous section in connection with continuous spacetime sheaves.

<sup>58</sup>See also (Breslav *et al.*, 1999, Raptis 2000a).

<sup>59</sup>Explicit matrix irreps of incidence algebras can be found in (Zapatrin, 1998).

<sup>60</sup>That is, the ‘product ideal’ of  $\mathcal{I}_p$  and  $\mathcal{I}_q$  in  $\Omega$ . Note in the parenthesis in (1) that primitive ideals (to be subsequently identified with ‘quantum points’) do not commute:  $\mathcal{I}_p \mathcal{I}_q \neq \mathcal{I}_q \mathcal{I}_p$ . In the next section this observation will prove to be invaluable both for the quantum and the local quantum time-directed interpretation of the structure of the  $\vec{\Omega}$ ’s, as well as for their comparison with the non-commutative  $C^*$ -quantales of Mulvey and Pelletier (2000).

<sup>61</sup>‘ $\rho$ ’, as an index of  $\mathcal{T}$ , indicates that it is the generating or ‘germ’ relation of this topology.

<sup>62</sup>See (Breslav *et al.*, 1999).

<sup>63</sup>Equivalently stated,  $\rho$  is the transitive reduction of the partial order relation ‘ $\rightarrow$ ’ (*ie*, formally  $\rho \equiv \xrightarrow{*}$ ) that defines the Sorkin poset topology of  $\Omega(P)$  (Raptis, 2000a).

<sup>64</sup>That is, give to its points  $\mathcal{I}_p$  the Rota topology  $\mathcal{T}_\rho(\mathbf{Spec} \Omega)$  generated by  $\rho$ .

<sup>65</sup>That is, the (kernels of equivalence classes of) irreps of  $\Omega$ .

discrete (*ie*, no linked pairs  $p \xrightarrow{*} q$ ), so that interesting topologies arise only when  $\Omega$  is non-commutative as we assumed above (Zapatrin, 1998, Breslav *et al.*, 1999, Raptis and Zapatrin, 2000)<sup>66</sup>. The reader should also note in (1) that homomorphic copies of  $\Omega$ , namely  $p$  and  $q$ , index the points of its primitive spectrum. By ‘inverting’ the symbols to  $\Omega(\mathcal{I}_p) \equiv \Omega_{p^{-1}(0)}$ , one gets a first impression of an  $\Omega$ -localization procedure over  $\mathbf{Spec}(\Omega)$  implicit in the above  $p$ -construction<sup>67</sup>.

Which brings us to the dual construction  $P \xrightarrow{\omega} \Omega(P)$  whereby to a finitary poset substitute  $P$  *à la* Sorkin (1991) one associates the incidence algebra  $\omega(P)$  according to the following steps:

- a. Represent  $p \rightarrow q$  arrows in  $P$ , by using the Dirac ‘ket-bra’ operator notation, as  $|p\rangle\langle q|$  in  $\Omega$  (Breslav *et al.*, 1999, Raptis and Zapatrin, 2000, Raptis, 2000a).
- b. Define  $\Omega(P)$  as a vector space over a field  $F$ <sup>68</sup> as follows

$$\Omega(P) = \text{span}_F\{|p\rangle\langle q| : p \rightarrow q \in P\} \quad (2)$$

- c. Define an associative product structure ‘ $\circ$ ’ on  $\Omega(P)$  as follows

$$|p\rangle\langle q| \circ |r\rangle\langle s| = |p\rangle\langle q|r\rangle\langle s| = \langle q|r\rangle \cdot |p\rangle\langle s| = \begin{cases} |p\rangle\langle s| & , \text{ if } q = r \\ 0 & \text{ otherwise.} \end{cases} \quad (3)$$

where ‘ $\cdot$ ’ is the usual  $F$ -scalar multiplication of vectors.

- d. Topologize  $\mathbf{Spec} \Omega(P)$  by first identifying the primitive ideals or points in it with subsets of the form

$$\mathcal{I}_p = \text{span}_F\{|q\rangle\langle r| : |q\rangle\langle r| \neq |p\rangle\langle p|\} \quad (4)$$

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<sup>66</sup>For instance, in the commutative case the Gel’fand topology is always discrete. Also, for finite dimensional algebras  $\Omega$  that we consider as being ‘physically pragmatic’ (Raptis and Zapatrin, 2000), another topology, the so-called Jacobson one, that can be imposed on  $\mathbf{Spec} \Omega$  is always discrete (Breslav *et al.*, 1999); hence our choice of the Rota topology for our finite dimensional, non-abelian Rota algebras  $\Omega$ .

<sup>67</sup>This remark will prove to be crucial in our subsequent identification of the curved finsheaves of gausets in (Mallios and Raptis, 2000) with primitive  $A$  or  $\Omega$ -schemes. The reader should also note that the name ‘spatialization’ given to the  $p$ -construction above (Zapatrin, 1998) is an appropriate one, because effectively one extracts from  $\Omega$  a topological space  $\mathcal{T}_p(\mathbf{Spec} \Omega)$ , which will later serve as the base ‘space’  $sp(\mathbf{Spec} \Omega)$  over which a primitive  $\Omega$ -scheme will be erected (see previous section).

<sup>68</sup>We may take  $F$  to be the field  $\mathbb{C}$  of complex numbers. This seems to enhance the quantum interpretation of the incidence algebras constructed by  $\omega$  (Raptis and Zapatrin, 2000, Raptis, 2000a).

and then  $\rho$ -relate them as in (1), thus give a topological space structure to  $\Omega(P)$ 's primitive spectrum  $\mathbf{Spec} \Omega(P) := \{\mathcal{I}_p\} \ (\forall p \in P)$  as follows

$$\mathcal{I}_p \rho \mathcal{I}_q := \mathcal{I}_p \mathcal{I}_q (\neq \mathcal{I}_q \mathcal{I}_p) \stackrel{\neq}{\subset} \mathcal{I}_p \cap \mathcal{I}_q \quad (5)$$

One can interpret this  $\omega$ -process as a lifting over  $\Omega(P)$ 's points<sup>69</sup> of local isomorphisms of  $\Omega(P)$ <sup>70</sup>. Thus, in a sense, the  $\omega$ -lifting of  $\Omega$  over  $P$  is a process ‘inverse’<sup>71</sup> to the  $p$ -soldering or localization of an  $\Omega$  on the poset  $P$  extracted from it *à la* Gel’fand as it was briefly described in the passage after (1).

Now we will comment briefly on the  $(\omega, p)$ -pair of ‘dual constructions’  $P(\Omega) \xrightarrow[\omega]{p} \Omega(P)$ .  $(\omega, p)$  may be regarded as a pair of contravariant functorial correspondences between the Sorkin poset category  $\mathfrak{P}$  of finitary topological posets and continuous (in the Sorkin topology) injections between them<sup>72</sup>, and the Rota poset category  $\mathfrak{R}$  of incidence algebras and surjective algebra homomorphisms, provided the posets in the former category are simplicial complexes or ‘nerves’ in the sense of (Alexandrov, 1956)<sup>73</sup>. The latter condition was shown to hold in (Raptis and Zapatrin, 2000)<sup>74</sup>, thus the Alexandrov-Sorkin poset category is indeed dual or ‘opposite’ to the Rota one<sup>75</sup>. It should be mentioned that the discovery of the dual functoriality of the  $(\omega, p)$ -pair in (Raptis and Zapatrin, 2000), which enables transitions between  $\mathfrak{P}$  and  $\mathfrak{R}$ , was essentially discovered first when Zapatrin (1998) showed that the composition of the  $(\omega, p)$

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<sup>69</sup>That is, its primitive ideals which are in 1-1 correspondence with the point-vertices of the ‘underlying’  $P$ .

<sup>70</sup>Actually, ‘homomorphic’ copies of  $\Omega(P)$  since the underlying points of  $P$  correspond to kernels (of equivalence classes) of irreducible homomorphic images (*ie*, irreps) of  $\Omega$ .

<sup>71</sup>Recall from footnotes 56 and 57 that the words ‘opposite’ or ‘dual’ could also be used instead of ‘inverse’ (see below).

<sup>72</sup>These are simply the partial order preserving maps  $f_{ij}$  between the finitary topological posets  $P_i$  in (Sorkin, 1991) that are partially ordered by refinement ‘ $\preceq$ ’. See also (Raptis and Zapatrin, 2000, Raptis, 2000b).

<sup>73</sup>This is the case for ‘nice’ topological spaces that one discretizes *à la* Sorkin (1991). Thus, the Sorkin poset category  $\mathfrak{P}$  should be properly called ‘the Alexandrov-Sorkin category’ and it consists of finitary posets or simplicial complexes, and injective poset morphisms or injective simplicial maps between them.

<sup>74</sup>See also (Zapatrin, 2000).

<sup>75</sup>In the sense that the functorial correspondences between them are in fact contravariant functors (Raptis and Zapatrin, 2000, Raptis, 2000b, Zapatrin, 2000). We will comment further on this duality later in this section when sheaf and scheme-theoretic localizations of Rota algebras will be involved, as well in the next section when the primitive finschemes of gausets will be organized into a quantum topos structure.

constructions  $p \circ \omega : P \xrightarrow{\omega} \Omega(P) \xrightarrow{p} P(\Omega(P))$  is an isomorphism between the poset  $P$  one starts with and the poset  $P(\Omega(P))$  resulting from the composite construction. One would normally expect that a  $P$ -morphism should correspond to an  $\Omega$ -homomorphism, but this is not always the case. As we just said, provided the original finitary topological poset  $P$  *à la* Sorkin (1991) is a simplicial complex *à la* Alexandrov (1956), this would indeed be the case<sup>76</sup>.

One can see more easily that the  $P$ s in  $\mathfrak{P}$  are simplicial complexes if one realizes that the incidence Rota algebras  $\Omega$  associated with them by the  $\omega$ -map above are in fact graded algebras. The latter is to say that every  $\Omega(P)$  splits into linear subspaces as follows

$$\Omega = \Omega^0 \oplus \Omega^1 \oplus \dots \quad (6)$$

where  $\Omega^0 = \text{span}\{|p\rangle\langle p| : \deg(|p\rangle\langle p|) = 0\}$ ,  $\Omega^1 = \text{span}\{|p\rangle\langle q| : \deg(|p\rangle\langle q|) = 1\}$ , and so on (Raptis and Zapatrin, 2000)<sup>77</sup>. In simplicial parlance, the relation  $[p \rightarrow q \in P] \xrightarrow[\omega]{p} [|p\rangle\langle q| \in \Omega]$  may be read as ‘ $p$  is a face of  $q$ ’<sup>78</sup>.

This graded algebra character of the  $\Omega(P)$ s also facilitates their interpretation as discrete differential manifolds (Dimakis and Müller-Hoissen, 1999, Raptis and Zapatrin, 2000, Breslav and Zapatrin, 2000). The  $\Omega^0$  subspace of  $\Omega(P)$ <sup>79</sup> is an abelian subalgebra of  $\Omega$  consisting of the algebra’s self-incidences<sup>80</sup> and the linear combinations thereof, and was coined ‘the space of stationaries’ in (Raptis and Zapatrin, 2000). The linear subspace  $\Omega^1$  of  $\Omega$  consists of linear combinations of the immediate arrows or links in  $P$  and was called ‘the space of transients’ in (Raptis and Zapatrin, 2000). Finally, the linear subspaces  $\Omega^i$  ( $i \geq 2$ ) of  $\Omega$  were called ‘the spaces of paths of length (or duration)  $i$ ’ in (Raptis and Zapatrin, 2000).

It is the categorical  $(\omega, p)$ -duality above between Sorkin’s finitary posets and their associated incidence Rota algebras that enhances the physical interpretation of the

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<sup>76</sup>Thus, it would be perhaps more accurate to call the Rota poset category  $\mathfrak{R}$  ‘the Rota-Zapatrin category’.

<sup>77</sup>Where ‘ $\deg(p)$ ’ is the degree or grade or cardinality of the simplex  $p$ , so that  $\deg(|p\rangle\langle q|)$  counts the difference in cardinality of  $p$  and  $q$  (*ie*, the number of vertices in  $P$  mediating between  $p$  and  $q$ ).

<sup>78</sup>See (Raptis and Zapatrin, 2000) for more details about the simplicial character of the  $\Omega(P)$ s.

<sup>79</sup>From now we will occasionally omit the poset  $P$  from which  $\Omega$  derives by the  $\omega$ -map above and simply write  $\Omega$ .

<sup>80</sup>That is, the identity arrows (reflexive relations)  $p \rightarrow p$  in the underlying  $P$  regarded as a ‘poset category’ (*ie*, the arrow product in the latter is simply the  $\circ$ -concatenation of the partial order arrows in  $P$  as in (3)).

latter as quantum spaces of discrete differential forms (Raptis and Zapatin, 2000)<sup>81</sup>. Furthermore, the inverse limit that the Alexandrov-Sorkin poset category  $\mathfrak{P}$  possesses when it is equivalently regarded as an inverse system of finitary topological posets (Sorkin, 1991, Raptis and Zapatin, 2000, Raptis, 2000b), which limit yields the classical  $C^0$ -manifold topology for spacetime as mentioned above, was interpreted for the contravariant Rota-Zapatin poset category  $\mathfrak{R}$ , now it also regarded as an inverse system or net, as Bohr's Correspondence Principle by Zapatin and this author (2000). Thus, a net of quantum spaces of discrete differential forms, at the physically non-pragmatic limit of infinite localization of spacetime events, 'decoheres' to a classical event living in a smooth continuum, the commutative algebra of its smooth coordinates, and the space of differential forms or covariant tensors tangent to it<sup>82</sup>.

It is also worth mentioning that it is exactly due to the richer algebraic structure of incidence algebras relative to that of posets that, first, the former have a sound quantum interpretation (Raptis and Zapatin, 2000, Raptis, 2000a), second, that they can be interpreted as discrete differential manifolds that reproduce at the inverse limit not only the continuous (*ie*,  $C^0$ ) topology of classical spacetime, but also its differential (*ie*,  $C^\infty$ -smooth) structure (Raptis and Zapatin, 2000, Raptis, 2000b, Mallios and Raptis, 2000), and third, that their localizations can be studied using powerful concepts, constructions and results from scheme theory<sup>83</sup>. This should be compared against the remarks in (Rideout and Sorkin, 2000) about the apparent sparseness of the locally finite posets' mathematical structure in the particular case that the latter represent causets (Bombelli *et al.*, 1987)<sup>84</sup>.

These last remarks bring us straight to the definition of qausets. Qausets have been defined as the causally and quantally interpreted incidence Rota algebras  $\Omega$  associated with Sorkin's finitary topological posets  $P$  (Raptis, 2000a) as described above. This definition was mainly inspired by Sorkin's (1995) insistence on a fundamental change of physical interpretation for the partial orders involved in his finitary

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<sup>81</sup>Briefly, stationaries are the discrete quantum analogues of the  $C^\infty$  coordinate functions of events in the smooth classical spacetime manifold, transients the discrete quantum correspondents of vectors cotangent to those classical events, and paths the discrete quantum analogues of higher degree differential forms on the smooth continuum. All in all,  $\Omega$  is a discrete and quantum analogue of the usual module of differential forms on the smooth continuum. The epithet 'quantum' refers to the interpretation of the linear structure (*ie*, '+') of the  $\Omega^i$ 's as 'coherent quantum superposition'.

<sup>82</sup>In particular,  $\Omega^0$  decoheres to the abelian algebra of the classical coordinates of spacetime events (usually taken to be the commutative  $C^*$ -algebra  $C^\infty(X, \mathbb{C})$  of holomorphic functions on  $X$ ),  $\Omega^1$  to the Lie algebra of covariant derivations at every point-event of the continuum, and the  $\Omega^i$ 's ( $i \geq 2$ ) to forms or covariant tensors of higher grade or degree or rank.

<sup>83</sup>This paper.

<sup>84</sup>See quotation in footnote 7.

substitutes for continuous topological spacetimes (Sorkin, 1991) ‘from topological to causal’, while their basic mathematical structure (*ie*, the locally finite poset) was essentially retained. This change in the physical semantics of locally finite posets had already resulted in the definition of causets (Bombelli, 1987, Sorkin, 1990*a,b*), only that qausets, being algebraic and not merely relational structures, can also afford a sound quantum interpretation as mentioned above (Raptis and Zapatrin, 2000, Raptis, 2000*a*). Thus, by emulating the sound semantics given to finitary topological posets in (Sorkin, 1991, Raptis, 2000*b*) as ‘locally finite approximations of the continuous, locally Euclidean topological relations between events in a bounded region  $X$  of a  $C^0$ -manifold spacetime  $M$ ’, qausets were subsequently interpreted as ‘locally finite and quantal replacements of the causal relations between events in a bounded region  $X$  of a (possibly curved) and smooth Lorentzian spacetime manifold  $M$ ’ (Mallios and Raptis, 2000)<sup>85</sup>. At the end of (Raptis, 2000*b*) it was explicitly anticipated that finsheaves of qausets could play an important role in formulating rigorously and entirely in algebraic terms a reticular, causal and quantal version of gravity.

Indeed, this possible application of curved finsheaves of qausets to represent a locally finite, causal and quantal version of Lorentzian gravity<sup>86</sup>, was the main theme in (Mallios and Raptis, 2000). In that paper it was explicitly shown how a straightforward translation of the Classical Equivalence Principle (CEP) of GR<sup>87</sup> in the reticular, causal and quantal algebraic realm of qausets, mandates that the latter be localized or gauged, hence curved. Thus we arrive swiftly at the most important ‘structural’ result about a sound mathematical model of the kinematics of a curved (thus dynamical) local quantum causal topology that was the main goal of this section:

**Fact.** The localized or gauged, thus curved, (principal) finsheaves of qausets presented in (Mallios and Raptis, 2000) are (principal)<sup>88</sup>  $A$ -finschemes<sup>89</sup> in the sense defined in the previous section.

Below we explain analytically this fact by basing our arguments on material taken

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<sup>85</sup>Again, we should emphasize the important change in the meaning of the incidence algebras involved: from quantum topological discretizations of ‘spatial’ nature (Zapatrin, 1998, Breslav *et al.*, 1999, Raptis and Zapatrin, 2000), to reticular quantum causal topologies of ‘temporal’ character (Finkelstein, 1988, Raptis, 2000*a*, Mallios and Raptis, 2000).

<sup>86</sup>A ‘finitary and causal quantum gravity’ so to speak.

<sup>87</sup>GR is formulated on a classical smooth spacetime continuum.

<sup>88</sup>The epithet ‘principal’ will be discussed shortly in connection with the local finitary spin-Lorentzian structure (gauge) symmetries of our primitive  $\vec{\Omega}$ -finschemes of qausets.

<sup>89</sup>Again, the term ‘finschemes’ meaning ‘finitary spacetime schemes’, and the algebras  $A$  being localized in it are the incidence Rota ones  $\Omega$  modeling qausets. Thus in our context, the  $A$ -schemes of the previous section are  $\Omega$ -schemes, issues of finitariness aside.

directly from (Mallios and Raptis, 2000), the definitions of the previous section and the brief presentation of posets and their incidence algebras above:

i) In (Mallios and Raptis, 2000) the base spaces for the finsheaves of qausets were initially taken to be the finitary topological poset substitutes  $P$  of a bounded region  $X$  of the curved smooth classical spacetime continuum  $M$ .

ii) Finsheaves of the incidence Rota algebras  $\Omega$  modeling quantum discretized manifolds as in (Raptis and Zapatrin, 2000) over the finitary topological posets  $P$  of i) were defined in the manner of (Raptis, 2000b). This was achieved essentially by recognizing that point-wise in the topological poset base space  $P$ <sup>90</sup> the  $\omega$ -map  $\omega : P \rightarrow \Omega(P)$  can be viewed as a local homeomorphic ‘lifting’ of a homomorphic copy of  $\Omega$  over  $P$ <sup>91</sup>. In turn, the latter can be thought of as defining  $\Omega(P)$  as an  $\Omega$ -sheaf over  $P$ , since  $\omega$ , by construction<sup>92</sup>, maps the germ  $\overset{*}{\rightarrow}$  of the Sorkin topology of  $P$  to the germ  $\rho$  of the Rota topology of  $\Omega(P)$ , thus it may be regarded as a local homeomorphism from the ‘base space’  $P$  to the ‘sheaf space’  $\Omega^\omega(P)$ <sup>93</sup>.

iii) Now, by topologically identifying  $P$  (and its Sorkin topology) with  $\mathbf{Spec} \Omega(P)$  (and its Rota topology)<sup>94</sup>, we take the latter to be the topological base space of the  $\Omega$ -sheaf  $\Omega^\omega(P)$  in ii). But the latter is nothing else than  $\Omega^\omega(\mathbf{Spec} \Omega)$ , thus it corresponds to the structure sheaf of the primitive  $\Omega$ -scheme  $\omega^s = (\mathbf{Spec} \Omega(P), \Omega^\omega(\mathbf{Spec} \Omega(P)))$  whose space  $sp(\omega^s)$  is clearly  $\mathbf{Spec} \Omega(P)$  equipped with the Rota topology<sup>95</sup>. It is crucial to note that the  $p$ -map that solders or localizes stalks in the structure sheaf space  $\Omega^\omega$  to points in the topological base space  $\mathbf{Spec} \Omega(P)$  is precisely the inverse of the  $\omega$ -map that ‘lifted’ the structure sheaf  $\Omega^\omega(\mathbf{Spec} \Omega(P))$  of the primitive  $\Omega$ -scheme  $\Omega^s$  and defined it as a local homeomorphism in ii). Formally this can be written as  $\omega = p^{-1}$  and it is in complete analogy to the definition of the sheaf map  $s$  and its inverse projection or localization or ‘soldering’ map  $\pi$  for the continuous sheaf  $\mathcal{S}(X)$  in the previous section<sup>96</sup>. That  $\omega = p^{-1}$  is, of course, a (local in  $\Omega^s$ ) consequence of

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<sup>90</sup>That is, locally over each of the point-vertices of the finitary poset  $P$ .

<sup>91</sup>See discussion following (5).

<sup>92</sup>See equations (1) and (5).

<sup>93</sup>Now the latter consists of independent stalks isomorphic (better, ‘homomorphic’ since reps of  $\Omega$  were used) to  $\Omega$  over  $P$ ’s points. Note that the superscript ‘ $\omega$ ’ of  $\Omega$  indicates ‘sheaf’ since it lifts individual vertices of the base poset to ‘germs’ in the stalks of  $\Omega^\omega$  over them by the local homeomorphism  $\omega$ .

<sup>94</sup>Since they are topologically indistinguishable (*ie*, homeomorphic) by the  $p$ -construction which is categorically dual to the  $\omega$  one.

<sup>95</sup>The superscript ‘ $s$ ’ of  $\Omega$  denoting ‘scheme’.

<sup>96</sup>See footnotes 21 and 57.

Zapatrin's theorem (1998) that  $P(\Omega(P)) \simeq P^{97}$ . Thus, our earlier calling the maps  $(\omega, p)$  inverse (or dual) of each other is literal in the primitive  $\Omega$ -scheme  $\Omega^s$ .

iv) Subsequently in (Mallios and Raptis, 2000) the authors re-interpreted the underlying topological base posets  $P$  as causets  $\vec{P}$  and their associated topological incidence algebras  $\Omega$  as gausets  $\vec{\Omega}^{98}$  as in (Raptis, 2000a). Thus we arrive at the notion of a primitive scheme  $\vec{\Omega}^s$  of gausets over a (possibly curved) finitary causal base space<sup>99</sup>. We must emphasize here that this mathematical structure is a sound model of a finitary, causal and quantal version of the CEP of GR<sup>100</sup>, as it was argued in (Mallios and Raptis, 2000). This model, the authors explained in that paper, is a sound model of the kinematical structure of QCD. Below we recall from (Mallios and Raptis, 2000) the basic arguments that led to the formulation of the FEP and its 'corollary', the FPGC<sup>101</sup>, for the curved finsheaves of gausets by presenting them in the new light of scheme theory.

If the underlying finitary causal base space is to be regarded as being 'curved', at least in the geometrical sense of this word, then these  $\vec{\Omega}$ -scheme-theoretic localizations can be physically interpreted as some kind of 'gauging of gausets' (Mallios and Raptis, 2000). This gauging is foreshadowed by the CEP of GR which, we recall, may be taken as saying that independent flat isomorphisms of Minkowski space  $\mathcal{M}$  are raised over every point-event of the curved classical spacetime continuum  $M$ . Similarly, in our primitive finschemes<sup>102</sup>  $\vec{\Omega}^s$ , independent flat gauset stalks are raised over every point-vertex of a curved finitary causal space which is then thought of as serving as a base space for the localization or soldering of these  $\vec{\Omega}$ -stalks of the structure sheaf of  $\vec{\Omega}^s$ .

As a result of this localization or gauging of gausets, the subsheaf morphisms  $d : \vec{\Omega}^i \rightarrow \vec{\Omega}^{i+1}$  effected by the flat<sup>103</sup> Kähler-Cartan differential  $d$  (Mallios, 1998, Raptis, 2000b, Mallios and Raptis, 2000), are 'gauged' into a 'curved' (*ie*, non-flat) connection operator  $\mathcal{D} = d + \mathcal{A}$  (Mallios and Raptis, 2000).  $\mathcal{D}$  is now a curved

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<sup>97</sup>Or in terms of  $p$  and  $\omega$ :  $p \circ \omega(P) \simeq P \Leftrightarrow p \circ \omega = \text{Id} \Leftrightarrow \omega = p^{-1}$ , 'Id' standing for 'isomorphism' or 'algebraic identity map' (see above).

<sup>98</sup>Note the arrow over their symbols that reminds one of the causal meaning of these structures.

<sup>99</sup>In fact, if like in (Mallios and Raptis, 2000) we assume that the causal base space is curved, we cannot identify it with a flat classical causet  $\vec{P}$ . Instead, it will be a flat transitive causet only locally (Mallios and Raptis, 2000).

<sup>100</sup>Called FEP in (Mallios and Raptis, 2000).

<sup>101</sup>The 'Finitary Principle of General Covariance'.

<sup>102</sup>That is, 'finitary spacetime schemes' as 'finsheaves' stands for 'finitary spacetime sheaves' (Mallios and Raptis, 2000).

<sup>103</sup>Because nilpotent (Dimakis and Müller-Hoissen, 1999, Mallios and Raptis, 2000).



finscheme morphism in the gauged  $\vec{\Omega}^s$ , and its non-flat part  $\mathcal{A}^{104}$  is a section of the ‘subscheme’<sup>105</sup>  $\vec{\Omega}^1$  (*ie*,  $\mathcal{A} \in \Gamma(U, \vec{\Omega}^1)$ )<sup>106</sup>, albeit, not a global section (Mallios, 1998, Mallios and Raptis, 2000)<sup>107</sup>. In (Mallios and Raptis, 2000) this non-existence of a global section for the curved finsheaves (here finschemes) of qausets was attributed to the non-transitivity of causality in the underlying curved causal base space due to gravity. Thus, the quantum causal topology encoded in the primitive finscheme of qausets  $\vec{\Omega}^s$  is not fixed once and forever as if the latter consists of a single qauset algebra  $\Omega$  over a single flat causet  $\vec{P}$ . Rather, over the points of the latter (now it regarded as being curved), local independent  $\vec{\Omega}$  isomorphs, ‘twisted’ or ‘warped’ relative to each other, are erected and the pattern of quantum causal connections between qauset elements of these stalks of  $\vec{\Omega}^s$  is a local dynamical quantum variable<sup>108</sup> (Mallios and Raptis, 2000).

This localization or gauging of the qausets in  $\vec{\Omega}^s$  and the resulting definition of the covariant derivative  $\mathcal{D}$  in the latter may be physically interpreted as follows: the dynamical variation of qausets from stalk to stalk in  $\vec{\Omega}^s$  may be attributed to the local (*ie*, point-wise) gauge freedom of selecting a qauset<sup>109</sup> from the independent stalks of the structure sheaf of  $\vec{\Omega}^s$ . If this ‘point-wise selection’ dynamical process for qausets is to respect the sheaf structure of  $\vec{\Omega}^s$ <sup>110</sup>, then it must be formulated categorically in terms of sheaf or, in our case, scheme morphisms<sup>111</sup>. The connection  $\mathcal{D}$  is conveniently such a scheme morphism that is readily seen to stitch the stalks of the structure sheaf of  $\vec{\Omega}^s$  in a way that respects the aforementioned ‘stalk-wise gauge freedom’ (Mallios,

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<sup>104</sup>This symbol for the ‘gauge potential’ of quantum causality (Mallios and Raptis, 2000) should not to be confused with same one used for symbolizing algebra sheaves in the previous section.

<sup>105</sup>A subscheme  $B$  of a given scheme  $C$  may be defined as a scheme whose structure sheaf is a subsheaf of that of  $C$ . It follows that the stalks of  $B$  are substructures of the stalks of  $C$ .

<sup>106</sup>All this follows from (6) which shows that the reticular and quantal analogue of the space  $\Omega^1$  of differential forms over the smooth spacetime continuum is a linear subspace of the qauset stalks  $\Omega$  of the structure sheaf of  $\vec{\Omega}^s$ .

<sup>107</sup>In the next section we are going to present and discuss in some detail a deep similarity between this non-existence of a global section for the gauge potential of quantum causality  $\mathcal{A}$  in the curved  $\vec{\Omega}^s$  and the non-existence of a global section in a certain topos of presheaves employed to model truth-valuations in quantum logic proper (Butterfield and Isham, 1998, 1999, Butterfield *et al.*, 2000).

<sup>108</sup>In our case this is the local quantum causality represented by  $\vec{\rho}$ . Note the arrow over the germ of the Rota topology  $\rho$  of  $\vec{\Omega}$  that reminds one of the latter’s causal interpretation.

<sup>109</sup>By a germ of the  $\mathcal{T}_{\vec{P}}$ -continuous sections of the structure sheaf of  $\vec{\Omega}^s$ .

<sup>110</sup>Physically speaking: to be in some sense ‘covariant’ with respect to the local quantum causal topology  $\mathcal{T}_{\vec{P}}$  encoded in the structure sheaf of  $\vec{\Omega}^s$  (Mallios and Raptis, 2000).

<sup>111</sup>For the definition of these, see previous section.

1998, Mallios and Raptis, 2000)<sup>112</sup>. Thus, the scheme-theoretic version of the sheaf-theoretic FPGC formulated in (Mallios and Raptis, 2000) reads: dynamical laws for the qausets in  $\vec{\Omega}^s$  must be equations between finscheme morphisms such as  $\mathcal{D}$ .

It follows that if one regards  $\vec{\Omega}^s$  as a principal primitive finscheme of qausets having for structure group  $\mathbf{G}$  a reticular version of the orthochronous Lorentz group  $L^+ = SO(1, 3)^\dagger$  (or its local isomorph  $SL(2, \mathbb{C})$ ) as in (Mallios and Raptis, 2000), one is able to interpret the FPGC above as some kind of local gauge invariance of the dynamics of qausets under  $\mathbf{G}$  (*ie*, as the local orthochronous Lorentzian relativity of the dynamics of the gauged qausets in the scheme). The primitive  $\mathbf{G}$ -scheme  $\mathcal{G}^s = (\mathbf{Spec} \vec{\Omega}, \mathcal{G}(\mathbf{Spec} \vec{\Omega}))$  associated with  $\vec{\Omega}^s$ <sup>113</sup> is the latter's local gauge symmetry structure and has as structure sheaf  $\mathcal{G}(\mathbf{Spec} \vec{\Omega})$  the localized or gauged reticular orthochronous Lorentz invariances of SR in accordance with the schematic version of the FEP above. Then  $\mathcal{A}$ , the non-flat part of  $\mathcal{D}$ , was seen in (Mallios and Raptis, 2000) to take values in the Lie algebra stalks of the structure sheaf of the finscheme  $\mathcal{G}^s$  associated with  $\vec{\Omega}^s$ <sup>114</sup>. This was seen to be in complete analogy to the classical smooth Lorentzian manifold case  $M$  on which (the kinematics of) GR may be effectively represented by a  $\mathbf{G}$ -bundle and a  $\mathbf{g}$ -valued gravitational connection 1-form  $\mathcal{A}$  on it, which, in turn, is a section of its associated bundle of modules of smooth Cartan forms on  $M$  (Mallios and Raptis, 2000). Finally, we mention again<sup>115</sup> that precisely by the way that curved finsheaves (here finschemes) of qausets were constructed, a dynamics for them expressed in terms of the sheaf (here scheme) morphism  $\mathcal{D}$  will be, by definition of the latter, 'gauge independent' if one physically interprets the open sets in the locally finite open covers of  $X$  with 'coarse local gauges' (*ie*, 'local coordinate patches or frames or laboratories of approximate measurements or localizations of the dynamical quantum causal relations between spacetime point-events') in  $X$  (Mallios, 1998, Mallios and Raptis, 2000). In this way the dynamics of qausets is independent of the background inert 'parameter base space'  $X$  whose sole purpose is to serve as a scaffolding for discretizing and subsequently soldering the fundamentally 'a-local'<sup>116</sup>

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<sup>112</sup>In this sense  $\mathcal{D}$ , which respects the local quantum causal topology (*ie*, the sheaf of qausets) generates a dynamics for qausets that is 'locally causal'. This is the finitary analogue of the classical differential locality (*ie*, infinitesimal local causality) of the gravitational spacetime continuum (Einstein, 1924, Mallios and Raptis, 2000).

<sup>113</sup>In (Mallios and Raptis, 2000) the  $\mathbf{G}$ -sheaf was also called 'adjoint to the  $\vec{\Omega}$ -sheaf'.

<sup>114</sup>That is, it was seen to be a  $so(1, 3)^\dagger \simeq sl(2, \mathbb{C})$ -valued reticular 1-form on the principal primitive  $\Omega$ -scheme  $(\vec{\Omega}^s, \mathcal{G}^s)$ .

<sup>115</sup>See footnote 12 in the introduction.

<sup>116</sup>The characterization of qausets as being alocal structures follows from the same interpretation given to their quantum discretized topological space relatives in (Raptis and Zapatin, 2000).

gausets, but in itself is of no physical significance (Mallios and Raptis, 2000). Such an independence of the gauset dynamics from an inert geometrical background spacetime is welcome for reasons discussed earlier in this paper and in more detail in (Mallios and Raptis, 2000)<sup>117</sup>.

At this point it must be noted that in view of the fact that the gausets in  $\vec{\Omega}^s$  coherently superpose with each other locally<sup>118</sup>, their local spin-Lorentzian structure symmetries in the adjoint principal primitive finscheme  $\mathcal{G}^s$  must also be quantum<sup>119</sup>. In turn, the latter entails that the finitary  $\mathbf{g}$ -valued non-flat spin-Lorentzian connections  $\mathcal{A}$  representing the gausets' dynamics will also be quantum variables; hence, as it was also anticipated in (Mallios and Raptis, 2000), perhaps a finitary and causal version of the usual 'covariant path-integral over connection space' quantization of Lorentzian gravity (Baez and Muniain, 1994) will be a good candidate for modeling QCD<sup>120</sup>.

Now one must realize that having a non-trivial  $\mathcal{D}$  on  $\vec{\Omega}^s$  may enable us to write a finitary, causal and quantal analogue of the left hand side of the classical Einstein equations of GR<sup>121</sup> by expressing the Einstein curvature tensor  $G_{\mu\nu}$  in terms of the reticular Lorentzian connection  $\mathcal{D}$ <sup>122</sup> as it is done in the usual connection-based gauge-theoretic approaches to GR<sup>123</sup> (Raptis, 2000c). This can be used then to represent the so-called vacuum Einstein equations<sup>124</sup> by equating with zero the left hand side of Einstein's equations (*ie*,  $G_{\mu\nu} = 0$ ;  $T_{\mu\nu} = 0$ ). However, in a non-vacuum situation (*ie*, in the presence of matter) it was until recently quite doubtful whether the right hand side of Einstein's equations<sup>125</sup> could be derived strictly from causal considerations and arguments. Rideout and Sorkin (2000) actually derive matter-like 'fields' from

<sup>117</sup>See also concluding physical remarks on possible curved finsheaves of gausets in (Raptis, 2000b.)

<sup>118</sup>In (Mallios and Raptis, 2000) this was the content of the Finitary Local Superposition Principle (FLSP) for the curved finsheaves of gausets.

<sup>119</sup>In the next section we will see how these local coherent quantum superpositions of gausets determine a quantum kind of subobject classifier, thus also a 'local quantum logic', for a topos organization of the  $\vec{\Omega}^s$ s. Essentially due to this, the resulting topos-like structure will be called a 'quantum topos'.

<sup>120</sup>See also remarks in section 5.

<sup>121</sup>Recall:  $G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu}$ .

<sup>122</sup>In turn, this expression may derive from varying with respect to  $\mathcal{A}$  a Lagrangian that is appropriately defined in terms of  $\mathcal{D}$ .

<sup>123</sup>The so-called Palatini formulation of GR or its recent spinorial formulation in terms of new variables (Ashtekar, 1986, Baez and Muniain, 1994). See also (Mallios and Raptis, 2000) for some comments on this in connection with the curved finsheaves of gausets, and also look at the discussion in the last section of the present paper.

<sup>124</sup>That is, in the absence of the matter tensor  $T_{\mu\nu}$ .

<sup>125</sup>The so-called energy-momentum tensor of matter  $T_{\mu\nu}$ .

causets, thus it seems reasonable, following Einstein's fundamental insight to equate the tensorial aspect of the spacetime geometry ( $G_{\mu\nu}$ ) with a similar tensor expression for the dynamical actions of matter ( $T_{\mu\nu}$ ), to represent matter actions in  $\vec{\Omega}^s$  again by appropriate finscheme morphisms between our gauged qausets (Raptis, 2000c)<sup>126</sup>.

In the next section we highlight the fundamentally non-commutative character of our incidence algebra localizations in  $\vec{\Omega}^s$  that model (the kinematics of) a curved (thus dynamical) local quantum causality and prepare the ground for a comparison between the non-commutative topology that  $\vec{\Omega}^s$  stands for and the  $C^*$ -quantale models of non-commutative topological spaces presented in (Mulvey and Pelletier, 2000). We will also compare our curved primitive finschemes of qausets against the 'warped' topos of presheaves of sets over the Stone space of a quantum lattice  $\mathcal{L}$  as presented recently in a series of papers on quantum logic proper (Butterfield and Isham, 1998, 1999, Butterfield *et al.*, 2000).

#### 4. NON-COMMUTATIVE ASPECTS OF PRIMITIVE $\vec{\Omega}$ -FINSCHEMES

First we describe briefly the Gel'fand spatialization method for extracting topological spaces from commutative and involutive (*ie*,  $*$ ) algebras  $A$  alluded to in the previous section<sup>127</sup>, then we pass to the non-commutative case of our particular interest here.

Let  $A$  be a commutative  $*$ -algebra. A natural or canonical way  $D$  to represent  $A$  is as a functional algebra on a topological space  $X$ <sup>128</sup>. One considers the following ideals in the representation algebra  $D(A)$  associated with every point  $x$  of  $X$

$$M_x := \{f \in D(A) : f(x) = 0\} \quad (7)$$

The factor algebra  $D(A)/M_x$  is seen to be 1-dimensional, hence it is simple. Thus, the kernels  $D^{-1}(M_x)$  of  $A$ 's 1-dimensional irreps  $D$ <sup>129</sup> are maximal ideals in  $A$  and, provided the underlying topological space  $X$  is reasonably 'nice'<sup>130</sup>, they are in 1-1

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<sup>126</sup>Thus in this way we will possess a reticular, causal and quantal analogue of Einstein's famous 'action-reaction' interpretation of the equations of GR which holds that geometry acts on matter in the form of gravity, and matter reacts back by curving geometry.

<sup>127</sup>See also (Zapatrin, 1998, Breslav *et al.*, 1999).

<sup>128</sup>In particular, Gel'fand worked with the  $*$ -representations  $D$  of commutative algebras in the abelian  $C^*$ -algebra  $C^0(X, \mathbb{C})$  of continuous complex valued functions on  $X$ . The latter is regarded as the 'standard representation' of commutative involutive algebras.

<sup>129</sup>Such  $D$ s are usually called 'characters'.

<sup>130</sup>We will not go into the technicalities of what kind of topological space is regarded as being 'nice'. For example, a locally Euclidean topological (*ie*,  $C^0$ ) manifold  $X$  is considered to be 'nice' and it

correspondence with the points of  $X$ . This correspondence is effectively used in the Gel'fand spatialization procedure whereby the points of  $X$  are first substituted by the maximal ideals of  $A$ , and then one imposes a suitable topology  $\mathcal{T}$  on the 'maximal spectrum'  $\text{Max } A$  of  $A$ . In this way, one 'extracts' in some sense a 'classical' topological space from a commutative algebra  $A$ <sup>131</sup>. It is also interesting to note in connection with the commutative case that this considering in (7) of the 'ideals of zeros' of the representation algebra  $D(A)$  point-wise in the representation domain  $X$  in order to define the point-set  $\text{Max } A$  to be subsequently topologized, is in complete analogy with how affine schemes are defined in the algebraic geometry of commutative polynomial rings over affine (Euclidean) space (Hartshorne, 1983, Shafarevich, 1994)<sup>132</sup>.

Now Gel'fand's method translates straightforwardly to the non-commutative case by passing from characters to equivalence classes of irreps, and accordingly, from maximal ideals to primitive ones. Then, there are many ways of imposing a topology on the points of the primitive spectrum  $\mathbf{Spec } A$  of a non-abelian algebra  $A$ . As we mentioned in the previous section, for our non-commutative and finite dimensional incidence Rota algebras  $\vec{\Omega}$  representing qausets (Raptis, 2000a), we chose the so-called Rota topology<sup>133</sup> instead of the more 'standard' ones of Jacobson or Gel'fand, because the latter reduce to the trivial, totally disconnected case<sup>134</sup> (Zapatrin, 1998, Breslav *et al.*, 1999, Raptis and Zapatrin, 2000). Then, primitive  $\vec{\Omega}$ -finschemes are defined by suitably localizing the  $\vec{\Omega}$ s as sheaves  $\vec{\omega}$  over  $\mathcal{T}_{\vec{\rho}}(\mathbf{Spec } \vec{\Omega})$ <sup>135</sup>.

The essentially non-commutative, thus quantal, character of the structure of  $\vec{\Omega}^s$  which models the kinematics of a gauged thus curved and dynamically variable local quantum causality  $\vec{\rho}$  (Mallios and Raptis, 2000), is explicitly manifested in the very

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is usually taken as the 'standard domain' for the functional representation of abelian algebras. The intimate relation between the locally Euclidean classical topological space  $X$  and the commutative structure of algebras that can be functionally represented on it will be discussed shortly in connection with the work of (Mulvey and Pelletier, 2000).

<sup>131</sup>Thus, the Gel'fand spatialization method may also be called 'algebra geometrization'.

<sup>132</sup>As we noted in section 2, the prime spectra of these commutative polynomial rings are first topologized according to the so-called Zariski topology  $\mathcal{T}_Z$ , and then sheaves of such rings  $R$  are erected over the Zariski topological space  $\mathcal{T}_Z(\mathbf{Spec } R)$  (see footnote 37).

<sup>133</sup>Which is generated by the relation  $\vec{\rho}$  on the point-events of  $\mathbf{Spec } \vec{\Omega}$  as (1) and (5) show. Note again here the arrow that is put over the generator  $\rho$  to remind one of its directly (local) causal meaning.

<sup>134</sup>That is, the discrete topology on  $\mathbf{Spec } \Omega$ .

<sup>135</sup>Which by the inverse 'soldering' local homeomorphism or sheaf  $\vec{\rho} = \vec{\omega}^{-1}$  corresponds to the curved causal base space  $\vec{P}(\vec{\Omega})$  whose constant transitive partial order causality  $\rightarrow$  is 'cut-off' by gravity to an intransitive dynamically variable germ of the causal topology  $\overset{*}{\rightarrow}$  (Raptis, 2000a, Mallios and Raptis, 2000).

definition of the latter. Apart from the fact that the stalks  $\vec{\Omega}$  of the structure sheaf  $\vec{\Omega}^{\vec{\omega}}(\mathbf{Spec} \vec{\Omega})$  of  $\vec{\Omega}^s$  are non-abelian Rota algebras, the very generating relation  $\vec{\rho}$  of quantum causality<sup>136</sup> is defined as a directed line segment (*ie*, an arrow)  $\xrightarrow{*}$  joining the point-events in  $\mathbf{Spec} \vec{\Omega}$  exactly ‘because’ a non-commutative relation<sup>137</sup> holds between them as (1) and (5) show.

In order to see more clearly the quantum physical interpretation of this non-commutativity<sup>138</sup>, we recall from (Mallios and Raptis, 2000) the quantum semantics given to the ideals in  $\vec{\Omega}(\vec{P})$  related by  $\vec{\rho}$ <sup>139</sup>.

First observe that when events  $p$  and  $q$  are immediately related by causality in  $\vec{P}$  (*ie*,  $p \xrightarrow{*} q$ ), the corresponding ideals  $\vec{\mathcal{I}}_p$  and  $\vec{\mathcal{I}}_q$  in  $\vec{\Omega}(\vec{P})$  are  $\vec{\rho}$ -related which, in turn, by the definition of the latter as in (1) and (5), is equivalent to

$$\vec{\mathcal{I}}_p \vec{\mathcal{I}}_q \neq \vec{\mathcal{I}}_q \vec{\mathcal{I}}_p \subsetneq \vec{\mathcal{I}}_p \cap \vec{\mathcal{I}}_q$$

Now the physical interpretation that can be given to  $\vec{\rho}$ -related primitive ideals in  $\vec{\Omega}(\vec{P})$  is as ‘transients’<sup>140</sup>, that is to say, elementary quantum dynamical processes of propagation of a quantum of causality<sup>141</sup> thus defining immediate reticular and quantal processes of ‘energy-momentum transfer’ between quantum events (Raptis and Zapatrin, 2000). On the other hand, the ‘position determinations’<sup>142</sup> of these  $\vec{\rho}$ -related events, namely, the ‘stationaries’  $|p\rangle\langle p|$  and  $|q\rangle\langle q|$  (Raptis and Zapatrin, 2000), are excluded by the very definition of the ‘quantum point-events’  $\vec{\mathcal{I}}_p$  and  $\vec{\mathcal{I}}_q$ <sup>143</sup> in  $\vec{\Omega}(\vec{P})$ , as (4) shows. Thus, the non-commutativity relation holding between quantum events

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<sup>136</sup>Occasionally we will call  $\vec{\rho}$  ‘local quantum causality’. Local quantum causality is the local dynamical variable defining by its dynamics the ‘observable quantum causal topology’  $\mathcal{T}_{\vec{\rho}}$  of  $\vec{\Omega}^s$  (Mallios and Raptis, 2000).

<sup>137</sup>That is,  $\mathcal{I}_p \mathcal{I}_q \neq \mathcal{I}_q \mathcal{I}_p$ .

<sup>138</sup>That is, interpret it as some kind of Heisenberg uncertainty relation built-in fundamentally at the germ-level of the (kinematical) structure for (dynamical) quantum causal topology that  $\vec{\Omega}^s$  stands for.

<sup>139</sup>See footnote 128 in (Mallios and Raptis, 2000).

<sup>140</sup>The elements of  $\vec{\Omega}^1$  as we saw in (6) of the previous section.

<sup>141</sup>Which may be called ‘causon’. We do not wish to use the name ‘chronon’ for the ‘elementary particle’ of quantum causality, because it has been used in Finkelstein’s Quantum Relativity theory of quantum spacetime and its dynamics, and there it stands for the elementary quantum of time (Finkelstein, 1988, 1996). Of course, one expects that causons and chronons are intimately related to each other in view of the close structural similarities between our QCD and Finkelstein’s curved quantum causal nets (Finkelstein, 1988, Raptis, 2000a, Mallios and Raptis, 2000).

<sup>142</sup>Or quantum acts of localization of the aforementioned causon.

<sup>143</sup>See subsequent comparison with the ‘spatial’ quantum points defined by Mulvey and Pelletier (2000).

in  $\vec{\Omega}(\vec{P})$  can be physically interpreted as an indeterminacy or uncertainty relation between quantum actions of localization (*ie*, determination of ‘position’) of a causon and their dual or complementary quantum actions of its momentous propagation defining the directed  $p \xrightarrow{*} q \equiv \vec{\mathcal{I}}_p \vec{\rho} \vec{\mathcal{I}}_q$  immediate quantum causal connections<sup>144</sup>. This may be viewed as a finitary and causal analogue of the usual kinematical Heisenberg uncertainty relations between the complementary time/position-energy/momentum  $(t, x)$ - $(E, p)$  observables of relativistic matter quanta propagating on a classical Minkowski continuum<sup>145</sup>.

It must be also emphasized that the non-commutativity relations between the quantum points in the primitive spectrum base space of  $\vec{\Omega}^s$  above reflect not only the ‘quantumness’ but also the ‘temporal directedness’ of the local quantum causality relations  $\vec{\rho}$  that bind them<sup>146</sup>, as follows

$$\begin{aligned} \vec{\mathcal{I}}_p \vec{\rho} \vec{\mathcal{I}}_q &\equiv \vec{\mathcal{I}}_p \vec{\mathcal{I}}_q (\neq \vec{\mathcal{I}}_q \vec{\mathcal{I}}_p) \stackrel{\neq}{\subset} \vec{\mathcal{I}}_p \cap \vec{\mathcal{I}}_q (= \vec{\mathcal{I}}_q \cap \vec{\mathcal{I}}_p) \Leftrightarrow p \xrightarrow{*} q \\ \vec{\mathcal{I}}_q \vec{\rho} \vec{\mathcal{I}}_p &\equiv \vec{\mathcal{I}}_q \vec{\mathcal{I}}_p (\neq \vec{\mathcal{I}}_p \vec{\mathcal{I}}_q) \stackrel{\neq}{\subset} \vec{\mathcal{I}}_q \cap \vec{\mathcal{I}}_p (= \vec{\mathcal{I}}_p \cap \vec{\mathcal{I}}_q) \Leftrightarrow q \xrightarrow{*} p \end{aligned} \quad (8)$$

To explain (8) in more detail, define  $\vec{P}^{\text{op}}$  to be the causet obtained from  $\vec{P}$  by reversing all of its arrows<sup>147</sup>. This ‘causality reversal’ unary map ‘op’ induces an algebra map  $\dagger : \vec{\Omega}(\vec{P}) \rightarrow \vec{\Omega}(\vec{P})^\dagger = \vec{\Omega}(\vec{P}^{\text{op}})$  whose action on an arbitrary element  $\omega$  of  $\vec{\Omega}(\vec{P})$  is given by

$$\omega = z_1 \cdot \omega_1 \circ \omega_2 + z_2 \cdot \omega_3 \circ \omega_4 + \dots \xrightarrow{\dagger} \omega^\dagger = z_1^* \cdot \omega_2^\dagger \circ \omega_1^\dagger + z_2^* \cdot \omega_4^\dagger \circ \omega_3^\dagger + \dots$$

where  $z_1^*$  and  $z_2^*$  are the complex conjugates of the complex coefficients  $z_1$  and  $z_2$  over which the quantum causal arrows  $\omega$  in  $\vec{\Omega}(\vec{P})$  coherently superpose (Raptis, 2000a)<sup>148</sup>, so that for  $\omega_1 = |p\rangle\langle q|$  in  $\vec{\Omega}(\vec{P})$  for example:  $\omega_1^\dagger = |q\rangle\langle p|$ .  $\omega_1^\dagger$  is called ‘the conjugate of  $\omega_1$ ’ and it is an element of ‘the Rota incidence algebra  $\vec{\Omega}(\vec{P})^\dagger$  conjugate to

<sup>144</sup>The energy-momentum-like ‘transients’ in  $\vec{\Omega}^1$  of (6).

<sup>145</sup>The reader may also like to refer to (Zapatrin, 1998, Breslav *et al.*, 1999) to see some possible connections between topological incidence Rota algebras (albeit, without a directly causal physical interpretation like our  $\vec{\Omega}$ s) and Noncommutative Geometry (Connes, 1994), especially when the latter is approached via  $C^*$ -algebraic non-commutative lattices as in (Balachandran *et al.*, 1996, Landi, 1997, Landi and Lizzi, 1997). See also remarks below in connection with the  $C^*$ -quantales of Mulvey and Pelletier (2000).

<sup>146</sup>See anticipating remarks in footnote 129 of (Mallios and Raptis, 2000).

<sup>147</sup>What is also commonly known as the poset category opposite to  $\vec{P}$ , hence the superscript ‘op’.

<sup>148</sup>Thus we assume indeed that the  $F$  in (2) is  $\mathbb{C}$ , as footnote 68 fixed.

$\vec{\Omega}(\vec{P})$ <sup>149</sup>. Note that the stationaries of  $\vec{\Omega}(\vec{P})$  in  $\vec{\Omega}^0$  are self-conjugate elements since  $\forall p \in \vec{P} : (|p\rangle\langle p|)^\dagger = |p\rangle\langle p|$ <sup>150</sup>.

Having defined  $\vec{\Omega}(\vec{P})^\dagger$ , we regard the opposite order of multiplication  $\mathcal{I}_q\mathcal{I}_p$  of quantum point-events in  $\vec{\Omega}(\vec{P})$  in the second line of (8)<sup>151</sup>, when ‘canonically’ mapped by  $\dagger$ , as actually defining ‘immediate quantum causal arrows’<sup>152</sup> in  $\vec{\Omega}(\vec{P})^\dagger$  over  $\vec{P}^{\text{op}}$ . This is because  $\vec{P}$  is a poset, thus if  $p \xrightarrow{*} q \in \vec{P}$  then  $q \xrightarrow{*} p \notin \vec{P}$ , since the transitive reduction  $\xrightarrow{*}$  of the partial order  $\rightarrow$  of  $\vec{P}$  is by definition antisymmetric<sup>153</sup>.

Thus the idea is to regard the product of two quantum points in the scheme  $\vec{\Omega}^s$  as being ‘locally directed’ in the sense that perhaps we should consider as being ‘physically significant’ the product of two primitive ideals in the scheme’s base space only in the same order that these points appear to be quantum causally  $\vec{\rho}$ -related in the latter. At the same time, we should regard the opposite order of multiplication of these quantum point-events as defining immediate quantum causal arrows of opposite direction in the conjugate scheme  $\vec{\Omega}^{s\dagger}$ <sup>154</sup>. In this way the non-commutative product between the quantum point-events in  $\vec{\Omega}^s$ ’s base space represents a distinction between the direction of quantum causality in the scheme. Thus the non-commutativity of the algebraic product in our primitive scheme of quantum causal sets reflects not only its quantal nature, but also its ‘temporal directedness’ which, we will hypothesize in the next section, lies at the heart of the fundamental<sup>155</sup> quantum time-asymmetry that is expected of the ‘true quantum gravity’ (Penrose, 1987).

Then we may call  $\vec{\Omega}^s$  ‘the curved quantum time-forward’ or ‘gauged future quantum causal topology’, and its conjugate  $\vec{\Omega}^{s\dagger}$  ‘the curved quantum time-backward’ or ‘gauged past quantum causal topology’<sup>156</sup>. Of course, it is a matter of convention or

<sup>149</sup> $\omega_1^\dagger$  is simply the Dirac ket-bra corresponding to the arrow  $q \rightarrow p$  in  $\vec{P}^{\text{op}}$  of opposite direction to the arrow  $p \rightarrow q$  in  $\vec{P}$ .

<sup>150</sup>This points to the direction of a possible ‘explanation’ of the reality ( $\mathbb{R}$ ) of the spacetime coordinates in the continuum Bohr correspondence limit of an inverse system of qausets (Raptis and Zapatin, 2000, Mallios and Raptis, 2000). However we are not going to discuss further this emergence of the  $\mathbb{R}$ -valuedness of the classical spacetime coordinates in the present paper.

<sup>151</sup>An order which is also held to be equivalent to the order or direction of the arrow  $q \xrightarrow{*} p$  in the base causet  $\vec{P}$ .

<sup>152</sup>Again, the arrows in the base causet  $\vec{P}$  may be regarded as being classical and their  $\vec{\omega}$ -correspondent qausets in  $\vec{\Omega}(\vec{P})$  as being quantum (Raptis, 2000a).

<sup>153</sup>That is,  $p \xrightarrow{*} q \Rightarrow q \not\xrightarrow{*} p$  (antisymmetry property of  $\xrightarrow{*}$ ).

<sup>154</sup>This is just the primitive scheme of qausets obtained by totally reversing the direction of the quantum arrows in the stalks of the structure sheaf of  $\vec{\Omega}^s$  and complex conjugating their  $\mathbb{C}$ -amplitudes (c-coefficients).

<sup>155</sup>Albeit, ‘kinematical’ in our case.

<sup>156</sup>Represented by the local quantum causal topological (dynamical) variables  $\vec{\rho}$  and  $\vec{\rho}^\dagger$ , respectively.



‘external choice’ which scheme one takes to be future and which past, but once this freedom of choice is fixed by a decision to adopt  $\vec{\Omega}^s$  or  $\vec{\Omega}^{s\dagger}$  for the kinematics of a dynamical local quantum causal topology, the dynamics of qausets in the respective schemes, which are represented by scheme morphisms as discussed earlier, will respect or preserve this ‘structural microlocal directedness of quantum causality’ or ‘kinematical local quantum arrow of time’. As we said in the last paragraph of the next section, we will return to discuss in some detail how this ‘local quantum arrow of time’ may be thought of as the characteristic feature of the kinematics of a time-asymmetric quantum gravity which then the dynamics of qausets (*ie*, our finitary and causal version of Lorentzian quantum gravity *per se*) that is formulated categorically in terms of scheme morphisms should conserve. Thus we will ‘justify’ Penrose’s (1987) claim that the true quantum gravity must be a time-asymmetric theory on the grounds that the QCD is time-asymmetric ‘because’ its kinematical structure, as encoded in  $\vec{\Omega}^s$ , is also locally quantum time-directed.

Now that we have discussed the fundamentally non-commutative character of the localized or gauged thus curved quantum causality modeled after the primitive finschemes  $\vec{\Omega}^s$ , we will briefly compare the latter with the recently proposed by Mulvey and Pelletier (2000) non-commutative topological spaces modeled after so-called  $C^*$ -quantales. We will first present abstract quantales in their dual role as non-commutative topological spaces and quantal logics, then we will discuss relevant elements from the (Mulvey and Pelletier, 2000) paper which presents a particular paradigm of quantales deriving from non-commutative  $C^*$ -algebras (especially by focusing on how these  $C^*$ -quantales may be viewed as non-commutative topological spaces similar to our  $\vec{\Omega}$ -finschemes of qausets), and finally we will comment briefly on the entries of table 1 attached at the end which summarizes the comparison between  $C^*$ -quantales and primitive  $\vec{\Omega}$ -finschemes.

An abstract quantale  $\mathfrak{Q}$  may be thought of as a lattice  $\mathcal{L}(\vee, \wedge)$  together with an ‘extra structure’  $\&$  which is an associative but non-commutative multiplication between its elements<sup>157</sup>. We write the quantale as the following triplet of structures  $\mathfrak{Q}(\vee, \wedge, \&)$ . For reasons to become transparent shortly, let us think of a quantale  $\mathfrak{Q}(\vee, \wedge, \&)$  as splitting into two lattice-like substructures

$$\mathfrak{Q}(\vee, \wedge, \&) = \begin{cases} \mathfrak{T}(\vee, \&) & , \text{ topological lattice} \\ \mathcal{L}(\vee, \wedge) & , \text{ logical lattice} \end{cases}$$

called ‘topological lattice’  $\mathfrak{T}(\vee, \&)$  and ‘logical lattice’  $\mathcal{L}(\vee, \wedge)$ , respectively.

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<sup>157</sup>For a more thorough treatment of quantales with various applications the reader is referred to (Rosenthal, 1990).

Let us call  $\wedge$  ‘the logical meet of the quantale’ and  $\&$  ‘the topological meet of the quantale’. The topological meet is non-commutative and distributes over arbitrary joins  $\vee$  of  $\mathfrak{Q}$ ’s elements, while the logical meet is commutative, but it is usually taken to be non-distributive over  $\vee$ . In this sense a quantale plays the following dual role as mentioned above: first, as an abstract topological space  $\mathfrak{T}(\vee, \&)$ ,  $\mathfrak{Q}$  may be thought of as a non-commutative generalization of ‘classical’, because commutative, topological spaces or locales  $\mathfrak{L}(\vee, \wedge)$  (Mac Lane and Moerdijk, 1992)<sup>158</sup>, and second, as a logical lattice  $\mathcal{L}(\vee, \wedge)$ ,  $\mathfrak{Q}$  may be thought of as an abstract quantum logic in the sense of Birkhoff and Von Neumann (1936)<sup>159</sup>. The particular  $C^*$ -quantales presented in (Mulvey and Pelletier, 2000) may be regarded as being both non-commutative topological spaces and quantum logics as we shall see below.

In (Mulvey and Pelletier, 2000) the quantales considered derive from non-commutative  $C^*$ -algebras and are interpreted as particular realizations of non-commutative generalizations of classical topological spaces (*ie*, realizations of abstract quantales). The central motivation for deriving quantales from non-abelian  $C^*$ -algebras is the possibility of an extension to the non-commutative case of the Gel’fand method of extracting classical topological spaces from commutative  $C^*$ -algebras that was briefly discussed in the beginning of this section. The motivating analogy with the commutative case is the following: as general ‘classical’ topological spaces, namely locales  $\mathfrak{L}(\vee, \wedge)$ <sup>160</sup>, are the results of applying the Gel’fand procedure to commutative  $C^*$ -algebras, due to which locales, in turn, may be called ‘commutative topological spaces’<sup>161</sup>, so quantales<sup>162</sup> may be obtained from applying an analogous Gel’fand procedure to non-commutative  $C^*$ -algebras<sup>163</sup>.

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<sup>158</sup>The usual example of a  $\mathfrak{L}(\vee, \wedge)$  being the  $\wedge$ -commutative lattice of open subsets of a given topological space with  $\vee$  and  $\wedge$  standing for the commutative set-theoretic union  $\cup$  and intersection  $\cap$  operations, respectively.

<sup>159</sup>That is,  $\mathfrak{Q}$  is an abstract non-distributive quantum lattice.

<sup>160</sup>Any complete distributive and  $\wedge$ -commutative lattice  $\mathfrak{L}(\vee, \wedge)$ , otherwise known as a complete Heyting algebra, is called a ‘locale’. As mentioned before, locales are viewed as generalizations of the lattices of open subsets of a given space  $X$  that classical topological spaces  $\mathcal{T}(X)$  are modeled after (Mac Lane and Moerdijk, 1992).

<sup>161</sup>As noted above in discussing abstract quantales, at the set-theoretic or topological level commutativity pertains to  $\cap$ ’s abelianess. In this sense, the set theory underlying locales (*ie*, classical topologies) is ‘classical’.

<sup>162</sup>The non-commutative extension or generalization of locales.

<sup>163</sup>Hence at the set-theoretic or topological level, the ‘quantal’ character of the resulting non-commutative topological spaces (from which they derive their name ‘quantales’), consists precisely in the definition of a non-commutative intersection-like operation  $\&$  between the elements of a quantale. Thus the set theory underlying quantales may be called ‘quantal set theory’. It would certainly be an interesting project to compare this non-commutative extension of classical set theory

The central issue in the quest for a sound quantal extension of locales *à la* Gel'fand is the question of what constitutes a point in the non-commutative  $C^*$ -algebraic context<sup>164</sup>. The Gel'fand procedure by which one can define 'quantum points' within non-commutative  $C^*$ -algebras is the main theme in (Mulvey and Pelletier, 2000). As we saw earlier, the Gel'fand method for 'spatializing' or constructing a topological space  $\mathcal{T}(X)$  from a commutative  $C^*$ -algebra  $A$  consists first in identifying the points of the space  $X$  to be extracted from  $A$  with the maximal ideals of  $A$ . Thus one identifies  $X$  with the spectrum  $\text{Max } A$ . Then one topologizes  $\text{Max } A$  by defining a suitable topology between its points. As Mulvey and Pelletier point out,  $\text{Max } A$  may be straightforwardly constructed as a locale by considering the 'propositional geometric theory' of closed prime ideals of the commutative  $A$  and the logic that this theory represents. In this case  $\text{Max } A$  is the distributive lattice of propositions of the logic of the theory commonly known as the Lindenbaum algebra of the theory.

Now, it seems natural to assume that for a non-abelian  $C^*$ -algebra  $A$ ,  $\text{Max } A$  in the previous paragraph should be substituted by  $A$ 's primitive spectrum  $\mathbf{Spec } A$  and a non-commutative quantal topology should be defined on its points. The latter as we saw previously in the context of our  $\tilde{\Omega}$ -schemes are no other than the kernels of (equivalence classes) of  $A$ 's irreps.

Indeed, as it is shown in (Mulvey and Pelletier, 2000), a 'good' spectrum  $\text{Max } A$  for defining a quantale in the non-abelian case is the set of closed two-sided ideals of  $A$  or, what it effectively amounts to the same, the space  $\{M_i\}$  of closed linear subspaces of  $A$ . The latter may be regarded as the 'quantum points' of the quantale. The topological lattice  $\mathfrak{T}(\vee, \&)$  structure defined on them is taken to be distributive and its joins ( $\vee$ ) and meets ( $\&$ ) are defined algebraically as follows

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and its resulting 'quantal' topology with Finkelstein's anticommutative (*ie*, Grassmannian) hence nilpotent extension of not only the classical, but also of the usual non-distributive Birkhoff-Von Neumann quantum logic, to a so-called 'quantum set theory' and its concomitant 'quantum spacetime topology' (Finkelstein and Hallidy, 1991, Finkelstein, 1996). In the same line of thought, it would certainly be worthwhile also to investigate whether our  $\tilde{\Omega}$ -schematic non-commutative topologies are particular instances of the abstract non-commutative topologies modeled after Grothendieck-type of schemes of associative non-abelian Polynomial Identity (PI) rings and algebras in what is called 'Non-Commutative Algebraic Geometry' (Van Oystaeyen and Verschoren, 1981, Van Oystaeyen, 2000*a, b*), since our incidence Rota algebras may be viewed as such non-commutative PI-rings (Fred Van Oystaeyen in private communication).

<sup>164</sup>Equivalently, the questions 'what is a quantum point ?' or 'what represents a point in a  $C^*$ -quantale ?'

$$\begin{aligned}
\bigvee_i M_i &= \overline{\sum_i M_i} \\
M_1 \& M_2 &= \overline{M_1 \cdot M_2} \\
M \& (\bigvee_i N_i) &= \bigvee_i (M \& N_i) \\
(\bigvee_i M_i) \& N &= \bigvee_i (M_i \& N)
\end{aligned} \tag{9}$$

where the join corresponds to the ‘closure of the linear span’, the meet to the ‘closure of the product’ and the meet left-right distributes over arbitrary joins.

One notices immediately that the topology defined on this quantale  $\text{Max } A$  is non-commutative exactly because the  $\&$  operation is not so. In return, the latter is not commutative, because it derives from the non-commutative product of the algebra  $A$ , as (9) shows. This non-commutativity should be compared directly with the non-commutative Rota topology defined between the ‘quantum points’<sup>165</sup> in the base primitive spectra of our primitive  $\vec{\Omega}$ -schemes which is generated by the relation  $\vec{\rho}$  that also depends crucially on the non-abelianess of the product of the incidence Rota algebras dwelling at the stalks of the structure sheaves of these schemes, as (1), (5) and (8) depict.

The reader can refer now to table 1 at the back which summarizes the comparison between our non-commutative  $\vec{\Omega}$ -schemes with Mulvey and Pelletier’s  $C^*$ -quantales. It must be said that we have not commented yet on the last two rows of table 1. That is to say, we have not commented on how ‘the underlying logic and geometry’ and ‘the physical interpretation’ of the two mathematical models compare against each other.

The underlying or ‘internal’ logic  $\mathcal{L}(\vee, \wedge)$  of  $C^*$ -quantales is seen to be ‘quantal’, analogously to how the internal logic of a locale is classical intuitionistic (Mac Lane and Moerdijk, 1992), in a rather strict technical sense that the reader can find in (Mulvey and Pelletier, 2000)<sup>166</sup>. Similarly, when we compare our  $\vec{\Omega}$ -schemes to Isham *et al.*’s topos-theoretic treatment of quantum logic proper shortly, we will argue that

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<sup>165</sup>In our case these are physically interpreted as quantum spacetime events as we saw earlier.

<sup>166</sup>Briefly, as the points of a locale constructed by the Gel’fand procedure from an abelian  $C^*$ -algebra  $A$  (*ie*,  $A$ ’s maximal ideals) correspond to the classical models of its underlying classical constructivistic logic (*ie*, ‘intuitionistic’ Heyting-Lindenbaum algebra), so the points of a quantale obtained by a similar Gel’fand procedure from a non-abelian  $C^*$ -algebra  $A$  represent the classical models of its propositional geometric theory within its ‘intrinsically quantal’ logic. This ‘intrinsically quantal’ characterization of the logic of the  $C^*$ -quantales considered in (Mulvey and Pelletier, 2000) may be ‘justified’ on the grounds that quantum (*ie*, Hilbert space) representations of the  $C^*$ -algebras were considered there, so that the underlying logical lattice  $\mathcal{L}(\vee, \wedge)$  of these Gel’fand-Hilbert  $C^*$ -quantales is a non-distributive quantum logic proper in the sense of Birkhoff and Von Neumann (1936). For more technical details, refer to (Mulvey and Pelletier, 2000).

the internal logic of a topos-like organization of our  $\vec{\Omega}^s$ s is locally ‘quantum’<sup>167</sup>. This is due to the local coherent quantum superpositions of gausets in the stalks of the structure sheaves of our  $\vec{\Omega}$ -schemes (Raptis, 2000*a*, Mallios and Raptis, 2000)<sup>168</sup>. Thus we will infer that the aforementioned topos of gausets is ‘locally quantalic’ indeed.

About the not localized or ungauged hence flat geometrical character of the  $C^*$ -quantales of Mulvey and Pelletier, as opposed to the localized or gauged thus curved geometry that our  $\vec{\Omega}$ -finschemes support, we bring the reader’s attention to the fact that the non-commutative  $C^*$ -algebras  $A$  considered in (Mulvey and Pelletier, 2000) are not localized (as sheaves) over their spectra like our  $\vec{\Omega}$ s are in  $\vec{\Omega}^s$ . By our remarks earlier in this section it follows that the  $C^*$ -quantales as presented in (Mulvey and Pelletier, 2000) are not suitable to model a variable (*ie*, dynamical) non-commutative topology, while our primitive finschemes of gausets, and the non-trivial spin-Lorentzian connections that they host, are<sup>169</sup>.

Finally, about the physical interpretation of the two mathematical models for non-commutative topology:  $C^*$ -quantales are constant and characteristically ‘spatial’ topologies defined between quantum ‘space’ points (Mulvey and Pelletier, 2000)<sup>170</sup>, as opposed to our primitive  $\vec{\Omega}$ -finschemes that model a dynamical local quantum causal topology<sup>171</sup>.

In closing this  $C^*$ -quantales *versus*  $\vec{\Omega}$ -finschemes comparison we mention that it would be very interesting to compare Table 1 at the end with a similar table presented in (Breslav *et al.*, 1999) where the Rota-algebraic approach to finitary spacetime topologies<sup>172</sup> is juxtaposed against an analogous  $C^*$ -algebraic non-commutative lattice approach due to Balachandran *et al.* (1996)<sup>173</sup>.

Now we wish to shed more light on the curved geometrical and quantum logical aspects of our primitive finschemes  $\vec{\Omega}^s$  by comparing them against the topos-theoretic models that were recently employed to describe the ‘warped’ and ‘non-objective’ na-

<sup>167</sup>In the sense of Birkhoff and Von Neumann (1936), that is to say, ‘non-distributive’.

<sup>168</sup>See footnote 118 in the previous section.

<sup>169</sup>For more on the curved character of a topos organization of our primitive finschemes of gausets the reader must wait until we compare our scenario with Isham *et al.*’s warped topos model of quantum logic proper.

<sup>170</sup>Hence by our remarks in the introduction, they are effective mathematical models for static, undirected, space-like connections between spatial, no matter if quantum, points.

<sup>171</sup>Hence by our remarks in the introduction, they are sound mathematical models for dynamically variable, directed, time-like, local connections between quantum spacetime events. This gauged or dynamical character of the gauset topologies in our finschemes will become even more transparent when we organize the latter into a topos-like structure shortly.

<sup>172</sup>Albeit, with not a directly causal interpretation for these topological structures.

<sup>173</sup>See also (Landi, 1997, Landi and Lizzi 1997).

ture of quantum logic which are the two main consequences of the Kochen-Specker theorem<sup>174</sup> of quantum logic (Butterfield and Isham, 1998, 1999, Butterfield *et al.*, 2000). First we will give a brief account of the results from (Butterfield and Isham, 1998, 1999) based on a concise exposition of these two papers by Rawling and Selesnick (2000), then we will cast our  $\tilde{\Omega}$ -schemes in such topos-theoretic terms, so that finally we will highlight this comparison by commenting on the entries of table 2 attached at the end.

So, following closely (Rawling and Selesnick, 2000) we let  $\mathcal{L}(\mathcal{H})$  be an orthomodular ortholattice, a so-called ‘quantum logic’<sup>175</sup>. At the ‘purely logical’ or propositional level quantum logics are distinguished from the lattices modeling classical logics in that they are fundamentally non-distributive (Birkhoff and Von Neumann, 1936). However, there is another equally fundamental and more ‘geometrical’ difference between quantum and classical logics: while the second are ‘flat’ in the sense that any Boolean algebra  $A$ <sup>176</sup> is isomorphic to the algebra of global sections of a sheaf of  $\mathbf{2s}$ <sup>177</sup> over the algebra’s Stone space  $\text{Spec } A$  (Selesnick, 1998), the first are ‘warped’ or ‘twisted’, or in a geometrical sense ‘curved’, relative to their Boolean substructures<sup>178</sup>. It is this difference that lies at the heart of the Kochen-Specker theorem of quantum logic as Butterfield and Isham point out.

More analytically, each Boolean subalgebra  $A$  in  $W$  represents a ‘classical window’ through which states of the quantum system may be ‘observed’.  $W$  is a poset category with the partial order ‘subset of’ relation  $A \subseteq B$  between two of  $\mathcal{L}(\mathcal{H})$ ’s Boolean subalgebras  $A$  and  $B$  being interpreted physically as ‘increasing the power of resolution in observing the quantum system’s states’<sup>179</sup>. The states that can be observed through a given  $A$  are the latter’s ‘spectral points’-elements of  $A$ ’s Stone space  $\text{Spec } A$ . In turn, the latter may be identified with the set of Boolean valuations of  $A$  into the ‘trivial’

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<sup>174</sup>Or commonly known as ‘paradox’ from the point of view of classical logic (Redhead, 1990).

<sup>175</sup>Usually this is supposed to be the lattice  $\mathcal{L}$  of closed subspaces of a Hilbert state space  $\mathcal{H}$  associated with a quantum system, hence the symbol  $\mathcal{L}(\mathcal{H})$ .

<sup>176</sup>Here we will not distinguish between classical logics and their corresponding Boolean algebras  $A$ .

<sup>177</sup> $\mathbf{2}$  is the algebra of the classical Boolean binary alternative  $\{0, 1\} \equiv \{\perp = \text{F}, \top = \text{T}\}$ . It is the ‘trivial’ Boolean subalgebra of every Boolean algebra  $A$ .

<sup>178</sup>That is, relative to their Boolean subalgebras in  $W[\mathcal{L}(\mathcal{H})] := \{A : A \subset \mathcal{L}(\mathcal{H})\}$ . It is a well established fact that the  $A$ s in  $W$  are generated by mutually compatible elements of  $\mathcal{L}(\mathcal{H})$  which, in more familiar physical terms, are eigenspaces of compatible (*ie*, commuting) ‘simultaneously measurable observables’ of the quantum system.

<sup>179</sup>That is,  $B$  enables one to ‘observe’ more finely or clearly the quantum system’s states than  $A$ .  $A$  provides a coarser view of the system’s states than  $B$ ; equivalently,  $B$  is finer than  $A$ .

Boolean algebra  $\mathbf{2}$ <sup>180</sup>. Thus each state of the quantum system that can be observed through  $A$  assigns to the elements of  $A$  a classical truth value in the Boolean binary alternative  $\mathbf{2}$ . If one abides to the realist existential and non-operational ideal that the states of a quantum system have some kind of ‘objective reality’, in the sense that they ‘exist’ independently of the acts or operations of observation of external ‘macroscopic’ or ‘classical’ observers, then they should be observable through each classical window  $A$  of  $\mathcal{L}(\mathcal{H})$  and, of course, such a view of them should be independent of the power of resolution that one employs to do so<sup>181</sup>. Such ‘objective reality states’ could then be modeled after ‘characteristic maps’ on each classical window  $A$ ,  $\chi_A : A \in W \rightarrow \text{Spec } A$  satisfying the restriction property

$$\chi_A = \chi_B|_A, (A \subseteq B); A, B \in W \quad (10)$$

Butterfield and Isham successfully observe that the main implication of the Kochen-Specker theorem is that no ‘global’ assignment  $A \mapsto \chi_A$  exists if  $\dim(\mathcal{H}) > 2$ . The epithet ‘global’ pertains to the fact that the correspondence

$$\mathbf{Spec} : W \rightarrow \text{Set} \quad (11)$$

between the poset categories  $W$  and  $\text{Set}$ <sup>182</sup> is a contravariant functor or presheaf<sup>183</sup> given ‘point-wise’ in the respective categories by

$$\begin{aligned} \text{object – wise : } \mathbf{Spec}(A) &= \text{Spec}(A) \in \text{Set} \\ \text{arrow – wise : } \mathbf{Spec}(A \subseteq B) &= \text{Spec}(A) \supseteq \text{Spec}(B) \in \text{Set} \end{aligned} \quad (12)$$

and as a presheaf it admits no global section of the type  $A \mapsto \chi_A$  (again, if  $\dim(\mathcal{H}) > 2$ ).

Now the category  $\text{Set}^{W^{opp}}$  of these presheaves is an example of a topos (Bell, 1988, Mac Lane and Moerdijk, 1992, Rawling and Selesnick, 2000), and it is well known

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<sup>180</sup>These valuations are homomorphic surjections of  $A$  onto its Boolean subalgebra  $\mathbf{2}$  of classical Boolean truth values.

<sup>181</sup>That is, even if  $A \subseteq B$ , one should be able to view the states of the quantum system equally clearly through either  $A$  or  $B$ .

<sup>182</sup> $\text{Set}$  consists of classical sets partially ordered by inclusion. Shortly we will see how the sets in  $\text{Set}$  can be interpreted as variable classical sets, thus they should be distinguished from the objects in the category  $\text{SET}$  which are usually interpreted as ‘constant’ classical sets (Bell, 1988, Selesnick, 1991, Mac Lane and Moerdijk, 1992).

<sup>183</sup>See (Mac Lane and Moerdijk, 1992, Raptis, 2000b).

that every topos has a terminal object  $1$  with respect to which global sections of its objects are defined. For  $\mathbf{Set}^{W^{opp}}$  in particular, arrows of the form  $1 \rightarrow \mathbf{Spec}$  are in 1-1 correspondence with global sections of  $\mathbf{Spec}$  which are easily seen to be assignments of the form  $A \mapsto \chi_A$  that do not exist by virtue of the Kochen-Specker theorem.

It must be said however that although global sections of  $\mathbf{Spec}$  do not exist, local ones do and correspond to consistent choices of valuations on certain Boolean subalgebras of  $\mathcal{L}(\mathcal{H})$ . Now the category  $\mathbf{Set}^{W^{opp}}$  qualifies as a topos, because among various structures in it there is an object  $\Omega$ , the so-called ‘subobject classifier’, relative to which subobjects of any given object  $\mathbf{Spec}$  of the topos are classified by injective arrows from the object into  $\Omega$ <sup>184</sup>. Local choices from or sections of  $\mathbf{Spec}$ <sup>185</sup> give rise to sub-presheaves of  $\mathbf{Spec}$ , thus lift to global presheaf morphisms of the following kind

$$\mathbf{Spec} \rightarrow \Omega \quad (13)$$

In the context of the Kochen-Specker theorem, Butterfield and Isham’s (1998, 1999) result can be stated as follows: if the target category  $\mathbf{Set}$  of the contravariant functor  $\mathbf{Spec}$  is identified with the classical Boolean topos  $\mathbf{SET}$  of constant classical sets whose subobject classifier  $\Omega$  is  $\mathbf{2}$ <sup>186</sup>, there is no global sheaf morphism of the sort depicted in (13)<sup>187</sup>. However, if one interprets the objects in  $\mathbf{Set}$  as ‘variable’ classical sets<sup>188</sup> like Butterfield and Isham do, one can re-express the negative result of the Kochen-Specker theorem in a positive way as follows

$$\mathbf{Spec} A \rightarrow \Omega(A) \quad (14)$$

Expression (14) may be interpreted as some sort of ‘localization or gauging of truth’ in  $W^{\mathbf{Set}^{opp}}$  whereby the trivial Boolean algebra  $\mathbf{2}$  is replaced by the  $A$ -dependent Heyting algebra<sup>189</sup>  $\Omega(A)$  of ‘generalized truth values’ that are localized or soldered on each object  $A$  in  $W$ . Thus, although global Boolean propositions or  $\mathbf{2}$ -valued valuations do not exist on the  $\mathbf{Spec}$  objects of  $W^{\mathbf{Set}^{opp}}$ , local ones, ‘varying per Boolean subalgebra  $A$  of  $\mathcal{L}(\mathcal{H})$ ’, abound at the expense of requiring these ‘truth-assignments’ to take values in  $A$ -dependent ‘generalized truth spaces’  $\Omega(A)$ <sup>190</sup>.

<sup>184</sup>See (Mac Lane and Moerdijk, 1992) for a detailed definition of  $\Omega$  with examples.

<sup>185</sup>That is, ‘local Boolean valuations’.

<sup>186</sup>The set of Boolean truth values.

<sup>187</sup>Object-wise, (13) defines ‘ $A$ -Boolean propositions’ of the form ‘ $\mathbf{Spec} A \rightarrow \mathbf{2}$ ’.

<sup>188</sup>Varying with respect to the Boolean subalgebras  $A$  of  $\mathcal{L}(\mathcal{H})$  in the base poset category  $W$ .

<sup>189</sup>That is, a complete distributive lattice, as we saw earlier in the context of quantales.

<sup>190</sup>Butterfield and Isham call these ‘context-dependent valuations’.



It is a standard result in topos theory that every topos has an ‘internal logic or language’ associated with it that is characteristically intuitionistic<sup>191</sup> (Lambek and Scott, 1986, Mac Lane and Moerdijk, 1992). This is reflected in the subobject classifier  $\Omega$  of the topos that is seen to be a complete Heyting algebra, or topologically speaking, a locale. Expression (14) is a paradigm of a deep affinity between ‘classical logical universes’, namely topoi, and ‘classical’ generalized topological spaces (*ie*, locales) which may be stated as follows: ‘topoi are (locally) localic’<sup>192</sup>.

In the case of the topos  $\text{Set}^{W^{opp}}$  associated with the quantum logic  $\mathcal{L}(\mathcal{H})$  (Butterfield and Isham, 1998, 1999) the aforementioned affinity between classical topoi and locales can be summed up into the following motto: ‘although quantum logic does not have a global notion of Boolean two-valued truth associated with its propositions like classical Boolean logic does, it has a local intuitionistic many-valued truth’<sup>193</sup> that still is of a ‘classical’ sort in the sense that the Heyting algebra that the  $A$ -local propositions of the quantum logic take their truth values still is a distributive lattice’. This feature of quantum logic may be called ‘local multi-valued realism (or classicism)’, or as Butterfield and Isham successfully coined it, ‘neo-realism’<sup>194</sup>.

In closing this presentation of the results from (Butterfield and Isham, 1998, 1999) via (Rawling and Selesnick, 2000), we mention that the topos  $\text{Set}^{W^{opp}}$  of variable sets over the base category  $W$  of Boolean subalgebras of a quantum lattice  $\mathcal{L}(\mathcal{H})$  is completely analogous to the topos  $\mathbf{Sh}(X)$  of sheaves of sets over a region  $X$  of Minkowski space  $\mathcal{M}$ <sup>195</sup> in which all the flat classical and quantum field theories are modeled (Selesnick, 1991). Indeed, in the latter paper it is argued how  $\mathbf{Sh}(X)$  may be regarded as a universe of continuously variable classical sets<sup>196</sup> varying over  $X$  which serves as a classical continuous background parameter space indexing this variation.

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<sup>191</sup>A so-called ‘Brouwerian type theory’ or ‘constructivistic logic’.

<sup>192</sup>Shortly we will dwell longer on this affinity between classical generalized logical universes, namely topoi, and classical generalized topological spaces, namely locales, in that we are going to present a topos-like organization of our primitive  $\tilde{\Omega}$ -finschemes of qausets as being a paradigm of structures called ‘quantum topoi’ that are the analogues of quantales much in the same way that classical topoi are the analogues of locales, namely, we will see that our quantum topos of qausets is ‘locally quantallic’. This was also mentioned in the context of  $C^*$ -quantales before.

<sup>193</sup>The points of the Heyting algebra  $\Omega(A)$  in (14) corresponding to ‘multiple truth values’. As we said earlier,  $\Omega(A)$  is a generalized truth space.

<sup>194</sup>Here we will call it ‘neo-classicism’ following mainly (Finkelstein, 1996) who identifies a version of ‘Platonic Realism’, called ‘Ontism’, with the basic philosophy underlying classical physics. See also (Redhead, 1990) for a discussion of the radical revision of the realist ideal of classical physics that the quantum revolution brought about.

<sup>195</sup> $\mathcal{M}$  regarded as a topological manifold (*ie*, a  $C^0$ -manifold).

<sup>196</sup>Like the elements of the target category  $\text{Set}$  in the topos  $\text{Set}^{W^{opp}}$ .

Selesnick emphasized that the classical sets may be regarded as being variable in the topos  $\mathbf{Sh}(X)$  exactly because its internal logic is a strongly typed non-Boolean intuitionistic Heyting algebra localized over or soldered on the points of  $X$ <sup>197</sup>. Selesnick, at the end of the (1991) paper, and mainly based on a reticular and quantal model for spacetime structure and its dynamics, namely Finkelstein’s ‘quantum causal net’, suggested that one should look for a quantum version of the classical spacetime topos  $\mathbf{Sh}(X)$  which could then be regarded as the fundamental structure in which to model a conceptually sound and pragmatically finite unification of GR and quantum theory (*ie*, ‘a finite quantum gravity’)<sup>198</sup>.

Below we present our primitive finschemes of gausets in topos-theoretic terms similar to the ones in which quantum logic was presented above, so that we can compare the respective structures and establish structural relationships between our curved gauset theory and the ‘warped’ quantum logic of Butterfield and Isham. Our ultimate aim will be to take the first steps in Selesnick’s quantum topos project<sup>199</sup>.

First we give the analogue in our theory of the presheaf expressions (11) and (12) above. As it was mentioned in the previous section, the Alexandrov-Sorkin poset category  $\mathfrak{P} = \{P_i, \preceq\}$  of finitary topological posets is ‘anti-equivalent’ or contravariant to the Rota-Zapatrin poset category  $\mathfrak{R} = \{\Omega_i, \succeq\}$  of topological incidence Rota algebras associated with them in the sense that there is a contravariant functor

$$\mathbf{Spec}^\omega : \mathfrak{P} \rightarrow \mathfrak{R} \quad (15)$$

from  $\mathfrak{P}$  consisting of finitary substitutes  $P_i$  and injective poset morphisms or refining relations (arrows) ‘ $\preceq$ ’ between them (Sorkin, 1991, Raptis, 2000b)<sup>200</sup> to  $\mathfrak{R}$  consisting of incidence Rota algebras  $\Omega_i$  associated with the  $P_i$ s of  $\mathfrak{P}$  and surjective algebra

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<sup>197</sup>In  $\mathbf{Sh}(X)$  the analogue of the base poset category  $W$  of  $\mathbf{Set}^{W^{opp}}$  is the poset category of open subsets of  $X$  partially ordered by set-theoretic inclusion ‘ $\subseteq$ ’.

<sup>198</sup>We may call Selesnick’s proposal ‘the quantum spacetime topos for finite quantum gravity project’ or ‘the quantum topos project’ for short.

<sup>199</sup>The possibility of approaching Selesnick’s quantum topos project via curved finsheaves of gausets was first noted at the end of (Mallios and Raptis, 2000).

<sup>200</sup> $\mathfrak{P} = \{P_i, \preceq\}$  was called ‘an inverse system or net of finitary substitutes’ in these papers and the refinement arrows were continuous injections (poset morphisms) from coarser to finer posets (see previous section). Note that these refinement relations ‘ $\preceq$ ’ between the topological posets in  $\mathfrak{P}$  are completely analogous to the partial order relations between the Boolean subalgebras of  $\mathcal{L}(\mathcal{H})$  in the base poset category  $W$  of  $\mathbf{Set}^{W^{opp}}$  that they too were physically interpreted as ‘coarsening the grain of observation’ (Rawling and Selesnick, 2000) or ‘increasing the power of resolution’ (Mallios and Raptis, 2000).

homomorphisms or coarsening relations (arrows) ‘ $\succeq$ ’ between them (Raptis and Zapatin, 2000)<sup>201</sup>. Point-wise in the respective categories the  $\mathbf{Spec}^\omega$  correspondence reads

$$\begin{aligned} \text{object – wise : } \mathbf{Spec}^\omega(P_i) &\equiv \omega(P_i) = \Omega_i(P_i) \in \mathfrak{R} \\ \text{arrow – wise : } \mathbf{Spec}^\omega(P_i \preceq P_j) &= \Omega_i \succeq \Omega_j \in \mathfrak{R} \end{aligned} \quad (16)$$

where in the first row we used the  $\omega$ -construction of the incidence algebra from its corresponding finitary poset that was presented in detail in the previous section. Clearly,  $\mathbf{Spec}^\omega$  is a presheaf in the same way that the correspondence  $\mathbf{Spec}$  in (11) was seen to be such. Furthermore, the object-wise  $\omega$ -correspondence between  $P_i$  and  $\Omega_i$  was seen to be a finitary spacetime sheaf in the sense of (Raptis, 2000b). Thus,  $\mathbf{Spec}^\omega$  is a presheaf of finitary spacetime sheaves of topological incidence algebras. It follows that if we evoke from (Raptis, 2000a) the ‘semantic processes’ of causalization and quantization for the posets of  $\mathfrak{P}$  and their corresponding incidence algebras in  $\mathfrak{R}$ , we arrive at the presheaf  $\mathbf{Spec} \xrightarrow{\vec{\omega}}$ <sup>202</sup> of finsheaves of qausets.

Due to the close structural similarities between the  $\mathbf{Spec}$  map of (11) and the  $\mathbf{Spec} \xrightarrow{\vec{\omega}}$  of (15), we infer that there is a topos-like organization of the  $\mathbf{Spec}$ s similar to how the  $\mathbf{Spec}$ s were organized into the ‘warped’  $\text{Set}^{W^{opp}}$ . We symbolize the resulting topos structure by  $\text{Qauset}^{\vec{\mathfrak{P}}^{opp}}$ <sup>203</sup>.

We propose that  $\text{Qauset}^{\vec{\mathfrak{P}}^{opp}}$  is a candidate for the quantum topos that Selesnick (1991) anticipated. We support our proposal on the following close parallels between it and  $\text{Set}^{W^{opp}}$ :

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<sup>201</sup>Write these Rota algebra surjections as  $\Omega_i \succeq \Omega_j$ . Parenthetically note that the  $(\omega, p)$ -duality between finitary substitutes and their incidence algebras mentioned in the previous section entails that whereas the inverse system  $\mathfrak{P}$  has an inverse limit space homeomorphic to a continuous manifold  $X$  (ie, the points of  $X$  are maximally localized at this limit) (Sorkin, 1991, Raptis, 2000b), its contravariant ‘direct system’  $\mathfrak{R} := \{\Omega_i(P_i), \succeq\}$  has a direct limit space that yields the stalks of  $S(X) \equiv \mathcal{C}^0(X)$ -the sheaf of continuous functions on  $X$  (ie, the points of  $S(X)$  are maximally localized at this limit; see section 2). These two categorically dual (‘opposite’) processes of localization of finitary posets (inverse limit) and their dual incidence algebras (direct limit) was first noticed at the end of (Raptis, 2000b); see also (Mallios and Raptis, 2000).

<sup>202</sup>From now on we will denote this as ‘ $\mathbf{Spec}$ ’ for short. Again, the arrow over  $\mathbf{Spec}$  reminds one of the causal meaning of the arrows of the objects of the  $\mathbf{Spec}$ -related  $\mathfrak{P}$  and  $\mathfrak{R}$  categories (which should also be written now as  $\vec{\mathfrak{P}}$  and  $\vec{\mathfrak{R}}$ ).

<sup>203</sup>Note that instead of ‘ $\vec{\mathfrak{R}}$ ’, we wrote ‘Qauset’ for the target poset category of the  $\mathbf{Spec} \xrightarrow{\vec{\omega}}$  objects of this topos just to pronounce its close similarity with  $\text{Set}^{W^{opp}}$  to be explored shortly.

(a) The ‘points’ in  $\text{Set}^{W^{opp}}$  are the Boolean subalgebras  $A$  of  $\mathcal{L}(\mathcal{H})$  (or the points in the corresponding Stone spaces  $\text{Spec } A$  mapped by the functor **Spec**). The sets in  $\text{Set}$  are regarded as varying over the base poset category  $W$  consisting of points and the latter may be viewed as the localization sites of the former.  $W$  may be thought of as the background base parameter space ‘indexing’ the variation of sets in  $\text{Set}$ . Similarly, the ‘quantum events’ in  $\text{Qauset}^{\vec{\mathfrak{P}}^{opp}}$  are the vertices of the underlying  $\vec{P}_i$ s in  $\vec{\mathfrak{P}}$  (more precisely, the quantum point-events in the primitive spectra **Spec**  $\vec{\Omega}_i(\vec{P}_i)$  by **Spec**). The qausets in  $\text{Qauset} \equiv \vec{\mathfrak{A}}$  are regarded as varying over the base poset category  $\vec{\mathfrak{P}}$  consisting of quantum events.  $\vec{\mathfrak{P}}$  may be thought of as the background parameter space ‘indexing’ the variation of qausets in  $\text{Qauset}$ .

(b) The interpretation of the sets in  $\text{Set}$  as variable entities in the topos  $\text{Set}^{W^{opp}}$  comes from the Kochen-Specker theorem which may be stated as follows: ‘the presheaf objects **Spec** in it admit no global section’, or stated in a positive way, that ‘the valuations or states  $\chi_A$  on the points  $A$  in the base poset category  $W$  are gauged or localized or have become context (point)  $A$ -dependent as  $\text{Spec } A \rightarrow \Omega(A)$ ’<sup>204</sup>. Similarly, we saw in the previous section that the qausets may be regarded as being dynamically variable entities due to their localization or gauging as primitive schemes over their primitive spectra **Spec**  $\vec{\Omega}_i(\vec{P}_i)$ <sup>205</sup>. Stated in a positive way the latter corresponds to the definition of non-flat connection operators  $\mathcal{D}_i = d_i + \mathcal{A}_i$  for each  $\vec{\Omega}_i^s$  object of the source category  $\vec{\mathfrak{P}}$  of the **Spec** objects of  $\text{Qauset}^{\vec{\mathfrak{P}}^{opp}}$ . These connections are  $\vec{\Omega}_i^s$ -scheme morphisms whose non-trivial part  $\mathcal{A}_i$  are sections of the  $\vec{\Omega}_i^1$  subschemes. As we noted in the previous section, that the  $\mathcal{D}_i$ s are non-flat means in turn that their non-trivial parts  $\mathcal{A}_i$  are local (*ie*, not global) sections of the  $\vec{\Omega}_i^1$ s.

(c) Let us dwell a bit on the last remarks in (b) above. We saw in connection with (13) how local sections of the presheaf objects **Spec** in  $\text{Set}^{W^{opp}}$  give rise to subpresheaves of **Spec** thus to global presheaf morphisms between the latter and another presheaf object structure  $\Omega$  in  $\text{Set}^{W^{opp}}$ -the topos’ subobject classifier-of the following sort

$$\mathbf{Spec} \rightarrow \Omega$$

or ‘locally’<sup>206</sup> as follows

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<sup>204</sup>The symbol for the subobject classifier should not be confused with that used for incidence Rota algebras. Context will make it clear what is meant by  $\Omega$ . In any case, qausets are symbolized by  $\vec{\Omega}$ s not  $\Omega$ s.

<sup>205</sup>Thus, now objects in  $\text{Qauset}^{\vec{\mathfrak{P}}^{opp}}$  are primitive finschemes  $\vec{\Omega}_i^s$  not finsheaves of qausets.

<sup>206</sup>That is, point-wise in the base category  $W$ .

$$\mathrm{Spec} A \rightarrow \Omega(A)$$

In complete analogy with  $\mathrm{Set}^{W^{opp}}$ , and since we saw in the previous section how  $\mathcal{A}_i$  is  $\mathcal{G}_i^s$ -valued<sup>207</sup>, local  $\mathcal{A}_i$  sections of the  $\vec{\Omega}_i^1$ s in  $\mathrm{Qauset}^{\vec{\mathfrak{P}}^{opp}}$  lift to global primitive  $G$ -finscheme morphisms of the following kind

$$\vec{\Omega}_i^s \rightarrow \mathcal{G}_i^s \tag{17}$$

or ‘locally’ in  $\vec{\Omega}_i^s$ <sup>208</sup>

$$\vec{\Omega}_i^{1s} \rightarrow \mathfrak{g}_i^s = \mathfrak{sl}(2, \mathbb{C})_i \tag{18}$$

By comparing the expressions (17) and (18) in  $\mathrm{Qauset}^{\vec{\mathfrak{P}}^{opp}}$  with (13) and (14) in  $\mathrm{Set}^{W^{opp}}$ , we infer that the analogue in the former topos of the localized or gauged Heyting algebra subobject classifier  $\Omega$  in the latter topos is a localized reticular and quantal version of the Lorentz-spin algebra  $\mathfrak{so}(1, 3)^\uparrow \simeq \mathfrak{sl}(2, \mathbb{C})$  of the orthochronous Lorentz group  $SO(1, 3)^\uparrow$  of local symmetries of Lorentzian gravity. Indeed, Selesnick (1994) working on Finkelstein’s quantum set theory and the quantum causal net dynamics based on it (Finkelstein, 1988, 1989, 1996)-a theory very similar to our reticular QCD as mentioned in (Raptis, 2000*a*, Mallios and Raptis, 2000)-found that a quantum version of the subobject classifier **2** of the topos SET of constant classical sets is  $\mathfrak{sl}(2, \mathbb{C})$ , and he interpreted the latter as the local relativity group of Finkelstein’s curvaceous quantum net<sup>209</sup>. Here too, it is understood that quantum causal sets coherently superpose in the stalks of the structure sheaves of the primitive  $\vec{\Omega}^s$ -finscheme objects of  $\mathrm{Qauset}^{\vec{\mathfrak{P}}^{opp}}$  which also means that their local structure or gauge symmetries

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<sup>207</sup>With  $\mathcal{G}_i^s$  the principal primitive finscheme of reticular and quantal orthochronous Lorentz ‘structure group symmetries’ of  $\vec{\Omega}_i^s$ . Locally,  $\mathcal{A}_i$  is supposed to take values in the reticular and quantal version  $\mathfrak{g}_i$  of the Lie algebra of the orthochronous Lorentz group  $\mathbf{G} = SO(1, 3)^\uparrow$  (*ie*,  $\mathfrak{sl}(2, \mathbb{C})_i$ ) (Mallios and Raptis, 2000).

<sup>208</sup>Recall from the previous section that ‘locally in  $\vec{\Omega}_i^s$ ’ means ‘for immediate quantum causal connections which are sections of the subfinscheme  $\vec{\Omega}_i^{1s}$  of  $\vec{\Omega}_i^s$ ’.

<sup>209</sup>This essentially shows how appropriate is Finkelstein’s choice to found the marriage of quantum with relativity theory (*ie*, Quantum Relativity, 1996) on a ‘Quantum Set Theory’, in that the quantization of the Boolean binary alternative ‘symmetry’ of classical sets yields directly the local relativity group of Lorentzian gravity.

encoded in their adjoint primitive finschemes  $\mathcal{G}_i^s$  must be quantal (Mallios and Raptis, 2000)<sup>210</sup>. Then we adjoin to every local object  $\vec{\Omega}_i^s$  ‘locally’ in the topos  $\mathbf{Qauset}^{\vec{\mathfrak{P}}^{opp}}$  a copy of the primitive finscheme of its reticular, causal and quantal structure symmetries  $\mathcal{G}_i^s$  as in (17)<sup>211</sup>, and regard the gauged or localized thus curved character of the  $\vec{\Omega}_i^s$ s as being represented by ‘global’ principal primitive finscheme morphisms, between the primitive  $\vec{\Omega}$ -finschemes of qausets and their adjoint or associated principal primitive  $\mathbf{G}$ -finschemes of their local structure symmetries, of the following kind *à la* (18)

$$\begin{aligned} \mathcal{D}_i : \vec{\Omega}_i^s &\rightarrow \mathcal{G}_i^s; \quad \mathcal{D}_i = d_i + \mathcal{A}_i \\ \mathcal{A}_i : \vec{\Omega}_i^{1s} \subset \vec{\Omega}_i^s &\rightarrow \mathbf{g}_i^s \subset \mathcal{G}_i^s \end{aligned} \quad (19)$$

If one gives the epithets ‘quantum logical’ to the topos  $\mathbf{Set}^{W^{opp}}$  and ‘quantum causal’ to  $\mathbf{Qauset}^{\vec{\mathfrak{P}}^{opp}}$ , one realizes that ‘local truth value of an  $A$ -proposition in the complete Heyting algebra  $\Omega(A)$ ’ in the quantum logical topos  $\mathbf{Set}^{W^{opp}}$ , translates to ‘local orthochronous Lorentz symmetry for the qausets in the stalk of the structure sheaf of  $\vec{\Omega}_i^s$  over the quantum spacetime event  $p_i$  of the object  $\vec{P}_i$  in the base category  $\vec{\mathfrak{P}}$  which event, in turn, corresponds to a primitive ideal of  $\vec{\Omega}_i(\vec{P}_i)$  in  $\mathbf{Spec} \vec{\Omega}_i(\vec{P}_i)$ ’.

(d) With the remarks in (a)-(c) in mind, one realizes that the variability of the classical sets in the target category<sup>212</sup>  $\mathbf{Set}$  of the presheaf objects  $\mathbf{Spec}$  in the warped quantum logical topos  $\mathbf{Set}^{W^{opp}}$  which is due to the localization or gauging or  $A$ -contextualization of classical Boolean (*ie*, **2**-valued) valuations on the  $A$ -propositions in the base category  $W$  of  $\mathbf{Spec}$ , translates to the dynamical variability of the qausets in the target category  $\mathbf{Qauset}$  of the objects  $\mathbf{Spec}$  in the curved quantum causal topos  $\mathbf{Qauset}^{\vec{\mathfrak{P}}^{opp}}$  which is due to the localization or gauging of the qausets’ reticular and quantal orthochronous Lorentz symmetries over the quantum spacetime events in the base category  $\vec{\mathfrak{P}}$  of  $\mathbf{Spec}$ .

It is important to note however that this ‘localization and concomitant warping of truth’ in  $\mathbf{Set}^{W^{opp}}$  results in a local logic or topology in the topos that is still classical; albeit, intuitionistic: the local generalized truth or topological space corresponding

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<sup>210</sup>This is our version of Finkelstein’s insight in Quantum Relativity (1996) that if a system is quantum, so must be its symmetries. See similar comments in the previous section in connection with footnote 118 about the FLSP and the local structure (gauge) symmetries of our finschemes of qausets.

<sup>211</sup>This is the analogue of (13) which holds in  $\mathbf{Set}^{W^{opp}}$ .

<sup>212</sup>As opposed to the base or source category  $W$ .

to the Heyting algebra or locale  $\Omega$  assigned locally to every Boolean subalgebra  $A$  of the quantum logic  $\mathcal{L}(\mathcal{H})$ , as (14) shows. In this sense the quantum logical topos  $\mathbf{Set}^{W^{opp}}$  is still classical, or better, ‘neo-classical’<sup>213</sup>. On the other hand, we have fundamentally assumed that the gausets dwelling in the stalks of the structure sheaves of the primitive finscheme objects of the topos  $\mathbf{Qauset}^{\vec{\mathfrak{P}}^{opp}}$  coherently superpose with each other, which implies that ‘the localization and concomitant curving of causality and its local orthochronous Lorentz symmetries’ in this topos are also in a strong sense quantum (Mallios and Raptis, 2000)<sup>214</sup>. Precisely for this we regard the quantum causal topos  $\mathbf{Qauset}^{\vec{\mathfrak{P}}^{opp}}$  a sound paradigm of a quantum topos. We further ‘justify’ this observation in (e) next.

(e) We must emphasize at this point that since the curved primitive  $\vec{\Omega}$ -finscheme objects of the target poset category  $\mathbf{Qauset}$  of the presheaf objects  $\mathbf{Spec}^{\rightarrow}$  of  $\mathbf{Qauset}^{\vec{\mathfrak{P}}^{opp}}$  are ‘quantalic’ *à la* Mulvey and Pelletier (2000) as we saw earlier, we may take the topos  $\mathbf{Qauset}^{\vec{\mathfrak{P}}^{opp}}$  to be an example of the ‘missing factor’ structure in the following analogy that has puzzled mathematicians for quite some time now

$$\frac{\text{topoi}}{\text{locales}} = \frac{?}{\text{quantales}} \quad (20)$$

it being understood that the elusive ‘quantum topos’ is the structure to replace the questionmark above<sup>215</sup>.

To give more evidence for how our  $\mathbf{Qauset}^{\vec{\mathfrak{P}}^{opp}}$  is a good example of such a quantum topos structure, we refer to a topos  $\mathbf{Set}^{\mathcal{V}^{opp}}$  analogous to the  $\mathbf{Set}^{W^{opp}}$  of (Butterfield and Isham, 1998, 1999, Rawling and Selesnick, 2000) presented in (Butterfield *et al.*, 2000) with the only structural difference between the two topoi being that the base category  $W$  in the second was replaced by a poset category  $\mathcal{V}$  of commutative  $C^*$  or Von Neumann algebras  $V$ . It is understood that if the non-commutative  $C^*$ -algebra

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<sup>213</sup>See footnote 193.

<sup>214</sup>For instance, the local logic in this topos of gausets must be ‘quantum’ in the general non-distributive lattice sense of Birkhoff and Von Neumann (1936), since the non-distributivity property of quantum logic is due to the coherent superpositions (quantum interference) of quanta—in our case, of gausets (Raptis, 2000a).

<sup>215</sup>Jim Lambek and Steve Selesnick in private communication. In short, we may take the analogy (20) as saying that as a classical topos is locally localic (topologically speaking) or locally intuitionistic/neo-classical (logically speaking) like the  $\mathbf{Set}^{W^{opp}}$  of Butterfield and Isham (1998, 1999) or the  $\mathbf{Sh}(X)$  of Selesnick (1991), so a quantum topos is (expected to be) locally quantalic; that is to say, locally it is a non-commutative topological lattice (topologically speaking) and a non-distributive logical lattice (logically speaking). Our  $\mathbf{Qauset}^{\vec{\mathfrak{P}}^{opp}}$  is locally quantalic in this sense.

$B(\mathcal{H})$  of bounded linear operators on the Hilbert space  $\mathcal{H}$  of a quantum system is considered, the objects in  $\mathcal{V}$  are the abelian Von Neumann subalgebras of  $B(\mathcal{H})$ <sup>216</sup>. Without giving more technical details from (Butterfield *et al.*, 2000), one can use the spectral theorem on  $B(\mathcal{H})$  to establish a 1-1 correspondence between the poset categories  $W$  and  $\mathcal{V}$ . Thus the presheaf objects in the topos  $\text{Set}^{\mathcal{V}^{opp}}$  were called ‘spectral presheaves’ and were symbolized by  $\Sigma$ . In  $\text{Set}^{\mathcal{V}^{opp}}$  the consequence of the Kochen-Specker theorem is that its  $\Sigma$  objects admit no global sections which, in turn, amounts to the definition of generalized (*ie*, Heyting-valued) and  $V$ -dependent (*ie*,  $V$ -contextualized) valuations on  $\mathcal{V}$  in complete analogy to the  $\text{Set}^{W^{opp}}$  case. Then,  $\text{Set}^{\mathcal{V}^{opp}}$ , like  $\text{Set}^{W^{opp}}$ , is seen to be (locally) localic or neo-classical<sup>217</sup>.

On the other hand, as Butterfield *et al.* (2000) remarked in constructing the spectra for the commutative  $*$ -algebras  $V$  in  $\mathcal{V}$ , if non-commutative  $C^*$ -algebras  $N$  were used as objects in the base category  $\mathcal{N}$ , their spectra would be quantales in the sense of (Mulvey and Pelletier, 2000)<sup>218</sup>. The natural conjecture is that the resulting topos-like structure,  $\text{Set}^{\mathcal{N}^{opp}}$  can be regarded as being ‘locally quantalic’ in contradistinction to  $\text{Set}^{W^{opp}}$  or  $\text{Set}^{\mathcal{V}^{opp}}$  which is logically ‘locally Heyting’ or topologically ‘locally localic’. Then perhaps it too could qualify as a possible candidate for a sound model of the aforementioned quantum topos similar to our  $\text{Qauset}^{\tilde{\mathfrak{P}}^{opp}}$ <sup>219</sup>. Since now both  $\text{Set}^{\mathcal{N}^{opp}}$  and our topos-like  $\text{Qauset}^{\tilde{\mathfrak{P}}^{opp}}$  allow for coherent quantum superpositions locally, they may be viewed as being ‘locally quantalic’<sup>220</sup>, hence both are good candidates for the quantum topos structure anticipated in (20). With these remarks we come to appreciate more the significance of adopting from (Finkelstein, 1988) the FLSP for

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<sup>216</sup>Interestingly enough, the maximal abelian subalgebras of non-commutative involutive algebras of quantum spacetime operations or actions are coined ‘classical frames’ in (Finkelstein, 1996, Selesnick, 1998), and the theory for the quantum structure and dynamics of spacetime called ‘Quantum Relativity’ may be equivalently named ‘Frame Relativity’. It is certainly interesting to investigate the extent to which a topos, like the  $\text{Set}^{\mathcal{V}^{opp}}$  above for example, is a sound mathematical model for implementing the basic ideas of Finkelstein’s Quantum Relativity theory on unifying the world’s quantum logic with its relativistic causal structure. Below we will give some more strong hints that our quantum topos  $\text{Qauset}^{\tilde{\mathfrak{P}}^{opp}}$  may be regarded as such a model.

<sup>217</sup>Just for this we will argue shortly that  $\text{Set}^{\mathcal{V}^{opp}}$ , in contradistinction to our  $\text{Qauset}^{\tilde{\mathfrak{P}}^{opp}}$ , can not be an example of the quantum topos that Selesnick (1991) anticipated to underlie Finkelstein’s curved quantum causal net.

<sup>218</sup>That is, in the non-commutative  $C^*$ -algebra case the analogue of a spectrum is the set of all closed two-sided ideals in the non-abelian  $N$ s. See table 1.

<sup>219</sup>Refer to table 1 to see in what sense our primitive  $\tilde{\Omega}$ -finscheme objects of  $\text{Qauset}^{\tilde{\mathfrak{P}}^{opp}}$  represent a localized or gauged version of the  $C^*$ -quantales of Mulvey and Pelletier (2000).

<sup>220</sup>That is to say, as being locally non-commutative topological lattices  $\mathfrak{T}(\vee, \&)$  and non-distributive quantum logic lattices  $\mathcal{L}(\vee, \wedge)$ .



the dynamically variable qausets in the stalks of the curved finsheaves in (Mallios and Raptis, 2000).

Of course, it is understood that the topos  $\text{Set}^{\mathcal{N}^{opp}}$  may be thought of as some kind of ‘Hilbert space bundle or sheaf’ on whose stalks (*ie*, Hilbert spaces) the non-abelian local  $C^*$ -observable algebras  $N$  in  $\mathcal{N}$  are represented. Evidently, the base category  $\mathcal{N}$  is no longer a poset category like the  $W$  in  $\text{Set}^{\mathcal{W}^{opp}}$ . It is significant to note however that such a sheaf-like structure of Hilbert spaces arises quite naturally in another topos-theoretic approach to quantum theory and quantum gravity, namely, Isham’s version of consistent histories (Isham, 1997)<sup>221</sup>. Moreover, finite dimensional Hilbert spaces associated with quantum lattices of consistent histories have recently been localized over the event-vertices of causets (Markopoulou, 2000)<sup>222</sup> in a manner very similar to how our qausets in  $\text{Qauset}^{\tilde{\mathfrak{P}}^{opp}}$  are localized over (curved) causal spaces  $\tilde{P}_i$ . The resulting structures were coined ‘quantum causal histories’. It is certainly worthwhile to try to relate our quantum topos of dynamically variable qausets  $\text{Qauset}^{\tilde{\mathfrak{P}}^{opp}}$  with the consistent quantum causal histories of Isham (1997) and Markopoulou (2000). The first step in this direction would be to view the finite dimensional irreps of our qausets as acting on Hilbert spaces in much the same way that finite dimensional Hilbert space irreps of topological incidence Rota algebras were studied in (Zapatrin, 1998, Breslav *et al.*, 1999). Such a study however has to be left for another paper.

Due to (a)-(d) above, we suggest that  $\text{Qauset}^{\tilde{\mathfrak{P}}^{opp}}$  is a good candidate for the quantum topos that Selesnick (1991) anticipated to underlie Finkelstein’s quantum set and net theory (Finkelstein, 1988, 1989, 1996), thus it can serve as a reasonable mathematical model for the kinematical structure of a reticular, causal and quantal version of Lorentzian gravity, as Mallios and Raptis (2000) also anticipated at the end of that paper.

Now the reader can refer to table 2 at the back for a comparison between the neo-classical quantum logical topos  $\text{Set}^{\mathcal{W}^{opp}}$  of Butterfield and Isham (1998, 1999) and our quantum causal quantum topos  $\text{Qauset}^{\tilde{\mathfrak{P}}^{opp}}$ . Conceptually, perhaps the most significant aspect of this comparison is that in  $\text{Qauset}^{\tilde{\mathfrak{P}}^{opp}}$  the analogue of ‘local truth value’ in  $\text{Set}^{\mathcal{W}^{opp}}$  is ‘local quantum causal symmetry’, so that it follows that the formal analogue in the quantum logical topos  $\text{Set}^{\mathcal{W}^{opp}}$  of local causality  $\xrightarrow{*}$  in the base poset category  $\mathfrak{P}$  of classical (albeit curved) causets is the classical logical implication structure ‘ $\Rightarrow$ ’. In view of the fact that it is quite problematic to define a sound implication connective in quantum logic (Rawling and Selesnick, 2000), and because

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<sup>221</sup>Chris Isham in private communication.

<sup>222</sup>Thus the base spaces of these finite dimensional Hilbert sheaves are again poset categories, albeit, with a directly causal interpretation of the partial orders involved and not as ‘refinement relations’.

$\xrightarrow{*}$  or its  $\vec{\omega}$ -equivalent  $\vec{\rho}$  has become a local dynamical quantum variable in Qauset $\vec{\mathfrak{P}}^{opp}$  via a non-flat  $\mathcal{D}$ , it follows that the troubles one has in defining a ‘global’ quantum implication may be not due to the non-distributive nature of quantum logic *per se*, but rather due to its warped or ‘twisted’ character as we saw above. If that turns out to be the case indeed, in complete analogy with the intransitive curved local causality  $\xrightarrow{*}$  in Qauset $\vec{\mathfrak{P}}^{opp}$  (Mallios and Raptis, 2000), we conjecture that  $\xrightarrow{*}^{223}$  is also intransitive<sup>224</sup>. The implication structure of quantum logic is ‘anomalous’ (Rawling and Selesnick, 2000) because the logic itself, in contradistinction to classical flat logic, is warped hence intransitive. This possibility however will have to be examined more thoroughly in a future paper (Raptis, 2000*d*).

In the final section of this paper we discuss the possibility that the fundamental quantum time-asymmetry, the so-called ‘quantum arrow of time’, expected of the ‘true quantum gravity’ (Penrose, 1987), is of a micro-local kinematical or structural and not of a dynamical nature as it is usually anticipated to be. We also argue that a quantum topos-theoretic model like our Qauset $\vec{\mathfrak{P}}^{opp}$  may prove to be a solid conceptual platform for a sound unified description of quantum logic and quantum gravity thus vindicate some early prophetic insights of Lawvere (1975) and Finkelstein (1969, 1979, 1996).

## 5. CONCLUDING REMARKS

‘*The true quantum gravity must be a time asymmetric theory*’ (Penrose, 1987)<sup>225</sup>. By this remark we understand in general that the fundamental asymmetry or directedness of time must be traced in a dynamical theory of quantum spacetime structure<sup>226</sup>.

In the Ashtekar formulation of self-dual canonical QGR (Ashtekar, 1986, Baez, 1994, Baez and Muniain, 1994) or in its equivalent covariant ‘path-integral over self-dual connection space’ approach (Baez, 1994, Baez and Muniain, 1994) for instance, Penrose’s imperative would translate into first formulating a time-asymmetric and quantum version of Einstein’s equations, then solving them to find time-asymmetric solutions. Indeed, Ashtekar (1986) first discovered asymmetric or chiral self-dual gauge connection spin-valued variables  $\mathcal{A}^+$  and re-wrote the Einstein equations of GR in terms of them, and then proceeded to a canonical quantization of this self-dual ‘spinorial’ GR. Subsequently, these chiral spinorial quantum Einstein equations

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<sup>223</sup>This is the local ‘neo-classical’ intuitionistic implication  $A$ -context-wise in  $\text{Set}^{W^{opp}}$ .

<sup>224</sup>It is only transitive locally (*ie*,  $A$ -wise).

<sup>225</sup>In fact, Penrose held that the true quantum gravity is a time asymmetric theory (see footnote 2), but in view of the fact that a conceptually cogent and finite quantum gravity has not been proposed yet, we have replaced the factual ‘is’ by the conjectural imperative ‘must be’.

<sup>226</sup>That is, when quantum spacetime is itself viewed as a dynamical variable.

were cast into a covariant ‘sum-over-histories or path-integral over the space of  $\mathcal{A}^+$  connections’ form, and then they were tried for solutions. Significant step in this direction was Rovelli and Smolin’s introduction of Wilson or  $\mathcal{A}^+$ -holonomy loops to the search for solutions of the Chern-Simons-Witten (CSW) path-integral version of the self-dual QGR of Ashtekar (Baez and Muniain, 1994)<sup>227</sup>. A series of subsequent discoveries of certain close affinities between the Rovelli-Smolin loop variables for quantum gravity and abstract mathematical objects called knots and links, resulted in the proposal that oriented or chiral knots may be solutions or ‘states’ for the self-dual quantum gravity in the CSW picture of the theory. Indeed, Kodama<sup>228</sup> showed how a state for CSW quantum gravity is a quantum version of anti-deSitter space. Shortly after it was appreciated how an oriented knot or link invariant, the so-called ‘Kauffman bracket’, was in fact a CSW state in the loop representation (Baez and Muniain, 1994). All in all, with the development of the loop approach to self-dual or chiral quantum gravity we have got a nice paradigm of Penrose’s ‘imperative’ above. However, one should note that the kinematical  $\mathcal{A}$ -connection space over which one defines the path-integral dynamics representing quantum gravity is that of the self-dual  $sl(2, \mathbb{C})$ -valued part  $\mathcal{A}^+$  of the gravitational gauge connection  $\mathcal{A}$  and not its ‘anti-chiral’ anti-self-dual part  $\mathcal{A}^-$  that takes values in  $sl(2, \mathbb{C})$ <sup>229</sup> (Baez and Muniain, 1994)<sup>230</sup>.

Current research in quantum gravity aside for a moment, it was a remarkable early achievement to find classical time-asymmetric solutions to the classical time-symmetric Einstein equations by employing a special coordinate system to chart the (topologically) undirected spacetime manifold<sup>231</sup> (Finkelstein, 1958)<sup>232</sup>. Since the smooth spacetime coordinates have a kinematical or structural rather than a dynamical role in GR<sup>233</sup>, Finkelstein’s discovery may be perceived as a kinematical or

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<sup>227</sup>The so-called ‘loop representation’ of self-dual quantum gravity.

<sup>228</sup>For a relatively complete list of references to the Rovelli-Smolin loop formulation of quantum gravity, the relation between knot theory and quantum gravity, the CSW covariant path-integral approach to quantum gravity, and to Kodama’s result, the reader is advised to look at (Baez and Muniain, 1994), as well as to refer to the back of various relevant papers in (Baez, 1994).

<sup>229</sup>The so-called ‘conjugate representation of the orthochronous Lorentz-spin algebra’.

<sup>230</sup>We will see shortly how this may be regarded as an analogue of the kinematical or structural local quantum time-asymmetry that we will propose in Qauset <sup>$\mathfrak{P}^{opp}$</sup> .

<sup>231</sup>As we said in the introduction, the locally Euclidean  $C^0$ -manifold topology of GR is based on undirected, reversible, two-way, space-like connections between its point-events (Finkelstein, 1988, 1991).

<sup>232</sup>The coordinate system alluded to above is known as the Eddington-Finkelstein coordinates.

<sup>233</sup>In the sense that in the classical theory of gravity the sole dynamical variable is the spacetime metric  $g$  (or its affine Christoffel connection  $\Gamma$  or the latter’s gauge-theoretic analogue-the spin-

structural explication of a ‘hidden’ time-asymmetry of classical gravity. The fact that time-asymmetric solutions exist for the time-symmetric Einstein equations of GR on a topologically undirected and spatial spacetime manifold, seemed quite paradoxical to the author at that time, since they appeared to violate the Principle of Sufficient Reason which in that case could be taken as holding that ‘time-symmetric causes must have time-symmetric effects’<sup>234</sup> (Finkelstein, 1958). Finkelstein resolved this apparent paradox by holding that it could be regarded simply as another consequence of the non-linear nature of gravity. Parenthetically, and in view of the structural or kinematical quantum time-asymmetry in the quantum topos  $\mathbf{Qauset}^{\tilde{\mathfrak{P}}^{opp}}$  that we are going to present shortly, we note that Finkelstein’s demonstration in (1958) that the exterior singularity of the Schwarzschild solution of the Einstein equations of GR is in fact a ‘unidirectional causal membrane’<sup>235</sup> entailed that the universe splits into two classes of particles: those whose gravitational fields allow only for future propagation of causal signals<sup>236</sup>, and those that allow only for a ‘past-propagating causality’<sup>237</sup>. In fact, at the end of the paper Finkelstein notes that “*in view of the delicate nature of the choice between the two classes, it is possible that the gravitational equations imply that all particles in one universe belong to the same class*”. Shortly we will essentially base the kinematical quantum arrow of time in  $\mathbf{Qauset}^{\tilde{\mathfrak{P}}^{opp}}$  on a similar ‘initial choice’ or ‘condition’ between  $\mathbf{Qauset}^{\tilde{\mathfrak{P}}^{opp}}$  and its quantum time-reverse  $(\mathbf{Qauset}^{\tilde{\mathfrak{P}}^{opp}})^{opp}$ .

Now we would like to reinstate the Principle of Sufficient Reason in gravity, as well as go against the current trend in quantum gravity research, and suggest that quantum gravity must be time-asymmetric simply ‘because’ the kinematical structure of quantum spacetime is so. In other words, we will argue that a local quantum arrow of time is already present or built-into our mathematical model for the kinematics of quantum spacetime, which time-asymmetric kinematical structure the dynamics (*ie*, ‘quantum gravity’) must then respect so as to ‘propagate and conserve the kinematical quantum arrow of time’.

Our proposal is really simple: we saw in the previous section how the quantum topos  $\mathbf{Qauset}^{\tilde{\mathfrak{P}}^{opp}}$  of the curved quantum time-forward, or gauged future quantum causal topologies  $\tilde{\Omega}^s$  is a sound model of the kinematics of a dynamically variable

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Lorentzian connection  $\mathcal{A}$ ), while the smooth real coordinates of the  $C^\infty$ -manifold are ‘gauged away’ by the general coordinate transformation group  $GL(4, \mathbb{R})$  or the diffeomorphism group  $\text{Diff}(M)$  thus implementing the Principle of General (Gauge) Covariance.

<sup>234</sup>The ‘causes’ in that case being Einstein’s field equations and the ‘effects’ their solutions.

<sup>235</sup>That is, causal influences in the gravitational field of a spherical point-particle can propagate strictly only to the future or only to the past.

<sup>236</sup>Which Finkelstein identifies with ‘particles’.

<sup>237</sup>Which Finkelstein identifies with ‘antiparticles’.

future local quantum causality  $\vec{\rho}$ . This then defined the ‘conjugate quantum topos’  $(\text{Qauset}^{\vec{\mathfrak{P}}^{\text{op}}})^\dagger = (\text{Qauset}^\dagger)^{(\vec{\mathfrak{P}}^{\text{op}})^{\text{opp}}}$  consisting of the curved quantum time-backward, or gauged past quantum causal topologies  $\vec{\Omega}^{s\dagger}$  is a sound model of the kinematics of a dynamically variable past local quantum causality  $\vec{\rho}^\dagger$ <sup>238</sup>.

So we have effectively obtained a binary alternative (choice) for the kinematical local quantum time-directedness which may be presented as follows

$$\text{Qauset}^{\vec{\mathfrak{P}}^{\text{opp}}} \text{ or } (\text{Qauset}^\dagger)^{(\vec{\mathfrak{P}}^{\text{op}})^{\text{opp}}} \quad (21)$$

This binary decision we may assume as having being made or ‘fixed’ upon the creation of the universe. In other words, we regard it as being determined by some kind of ‘initial condition’<sup>239</sup> for the dynamics of local quantum causality of the following sort:

*The initial kinematical quantum causal structure is  $\text{Qauset}^{\vec{\mathfrak{P}}^{\text{opp}}}$ .*

Our proposal is in complete formal analogy to Penrose’s (1987) ‘*Weyl Curvature Hypothesis*’ for the fundamental quantum time-asymmetry of the true quantum gravity which may be stated as follows:

*The Weyl curvature tensor of gravity*<sup>240</sup> *was initially zero:*  $W|_{t_0=0} = 0$ .<sup>241</sup>

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<sup>238</sup>Recall from the previous section that  $\vec{\rho}^\dagger$  may be thought of as the past local dynamical quantum causality relation holding between quantum spacetime events as depicted in the second line of (8). It is certainly interesting in this context to mention the remark at the end of (Mulvey and Pelletier, 2000) that if the propositional interpretation of an element ‘ $a$ ’ of a Gel’fand-Hilbert  $C^*$ -quantale is as an action on (the Hilbert space states of) a quantum system, then its adjoint or involute ‘ $a^*$ ’ is interpreted as an action in reverse time. Thus our interpretation of the elements of  $\text{Qauset}^{\vec{\mathfrak{P}}^{\text{opp}}}$  and  $(\text{Qauset}^\dagger)^{(\vec{\mathfrak{P}}^{\text{op}})^{\text{opp}}}$  as local and curved quantum causal actions opposite in (quantum) time is well justified. Of course, as we saw in the previous section and as it is written in table 1, the fact that our qauset algebras  $\vec{\Omega}_i$  are not involutive (*ie*, they are not closed under  $\dagger$ ) like their  $C^*$ -quantale relatives, is precisely what allows for this fundamental separation of qausets (and their quantum topoi) into ‘two universes of local curved quantum causal actions’, one future ( $\text{Qauset}^{\vec{\mathfrak{P}}^{\text{opp}}}$ ), the other past ( $(\text{Qauset}^\dagger)^{(\vec{\mathfrak{P}}^{\text{op}})^{\text{opp}}}$ ), much like the disjoint future and past unidirectional causality universes in (Finkelstein, 1958) that we saw earlier (see also below).

<sup>239</sup>The words ‘choice’, ‘decision’ or ‘constraint’ could also be alternatively used as synonyms to ‘condition’.

<sup>240</sup>The part  $W$  of Einstein’s tensor  $G$  (or equivalently Riemann’s  $R$ ) that measures the entropy of the gravitational field.

<sup>241</sup>In this way Penrose also ‘explains’ the origin of the so-called ‘thermodynamic arrow of time’. Our considerations however are entirely quantum, not statistical/thermodynamical (Raptis and Zapatin, 2000, Mallios and Raptis, 2000).

It follows that the ‘categorical dynamics’ of the variable local quantum causality  $\vec{\rho}$  in the quantum topos  $\mathbf{Qauset}^{\vec{\mathfrak{P}}^{opp}}$ <sup>242</sup> formulated entirely in terms of primitive finscheme morphisms will preserve the ‘micro-local future-directedness’ of  $\vec{\rho}$  thus it will in a sense ‘conserve’ the kinematical micro-local initial quantum arrow of time. This may be loosely called ‘the principle of conservation of the initial kinematical micro-local quantum time-asymmetry by quantum gravity’<sup>243</sup>.

At this point we must emphasize that if the  $\vec{\Omega}^s$  objects of  $\mathbf{Qauset}^{\vec{\mathfrak{P}}^{opp}}$  have a finitary and quantal version of  $sl(2, \mathbb{C})$  as local structure or relativity group<sup>244</sup>, then the  $\vec{\Omega}^{s\dagger}$ s in the ‘quantum time-reverse’ or conjugate topos  $(\mathbf{Qauset}^\dagger)^{(\vec{\mathfrak{P}}^{op})^{opp}}$  are expected to have a locally finite and quantal version of  $\overline{sl(2, \mathbb{C})}$ <sup>245</sup> as their local relativity group structure. The qausets in the  $\vec{\Omega}^s$ s of  $\mathbf{Qauset}^{\vec{\mathfrak{P}}^{opp}}$  may be thought of as ‘left-handed’ spinors transforming as vectors under the fundamental or regular irrep  $S$  of  $sl(2, \mathbb{C})$ , while the qausets in the  $\vec{\Omega}^{s\dagger}$ s of  $(\mathbf{Qauset}^\dagger)^{(\vec{\mathfrak{P}}^{op})^{opp}}$  may be thought of as ‘right-handed’ spinors transforming as vectors under the conjugate irrep  $S^*$  of the spin-group corresponding to  $\overline{sl(2, \mathbb{C})}$ <sup>246</sup>. Thus, the aforementioned micro-local quantum time-asymmetry may be alternatively stated as follows:

*Only left-handed causons<sup>247</sup> exist in the quantum spacetime deep<sup>248</sup>.*

Interestingly enough, Finkelstein (1988) and Selesnick (1994, 1998) have come to the same conclusion about the ‘chirality’ or ‘handedness’ or ‘micro-local quantum time-asymmetry’ of the dynamical spinorial spacetime quanta<sup>249</sup> of Finkelstein’s quantum

<sup>242</sup>That is, a locally finite, causal and quantal version of Lorentzian gravity (Mallios and Raptis, 2000). See also section 4.

<sup>243</sup>Again, ‘quantum gravity’ being perceived in our theoretical scenario as the finitary dynamics of local quantum causality  $\vec{\rho}$  (Mallios and Raptis, 2000).

<sup>244</sup>See (Mallios and Raptis, 2000) and previous section.

<sup>245</sup>This is the complex conjugate of the Lie algebra  $sl(2, \mathbb{C})$ .

<sup>246</sup>Note that  $S$  and  $S^*$  are inequivalent irreps. Also, since  $sl(2, \mathbb{C})$  is isomorphic to the orthochronous Lorentz Lie algebra  $so(1, 3)^\dagger$ ,  $\overline{sl(2, \mathbb{C})}$  may be regarded as being isomorphic to the ‘antichronous’ (ie, time-reverse) Lorentz Lie algebra  $so(1, 3)^\dagger$ .

<sup>247</sup>Recall from the previous section that we called ‘causons’ the dynamical quanta of local quantum causality  $\vec{\rho}$ .

<sup>248</sup>It follows that if  $\mathbf{Qauset}^{\vec{\mathfrak{P}}^{opp}}$  stands for the kinematics of the  $\vec{\rho}$  causons,  $(\mathbf{Qauset}^\dagger)^{(\vec{\mathfrak{P}}^{op})^{opp}}$  stands for the kinematics of the  $\vec{\rho}^\dagger$  ‘anticausons’. The latter are viewed as the quantum relativistic antiparticles of causons that dynamically propagate in the quantum time-reverse (ie, antichronous) direction of the orthochronous causons, thus they have ‘local (gauge) symmetry’  $\overline{sl(2, \mathbb{C})} \simeq so(1, 3)^\dagger$ .

<sup>249</sup>The so-called ‘chronons’.

causal net: they are left-handed or future-directed quanta<sup>250</sup>. We also note here that this initial kinematical choice for the future quantum causal universe  $\text{Qauset}^{\vec{\mathfrak{P}}^{opp}}$  populated exclusively by causons, and against the past quantum causal universe  $(\text{Qauset}^\dagger)^{(\vec{\mathfrak{P}}^{op})^{opp}}$  inhabited solely by anti-causons, is completely analogous to the time-asymmetric or ‘causally unidirectional’ classical gravitational fields of point particles (and their antiparticles) mentioned earlier in connection with (Finkelstein, 1958).

Now the reader can compare this view of ours about the ‘origin’ of the fundamental quantum time-asymmetry expected of the true quantum gravity with the aforementioned path-integral over self-dual  $sl(2, \mathbb{C})$ -valued gravitational gauge connections  $\mathcal{A}^+$  of Ashtekar *et al.* The bottom line is that according to our view the quantum gravity represented by the latter scenario is time-asymmetric ‘because’, structurally or kinematically speaking, only chiral left-handed gravitational spin variables  $\mathcal{A}^+$  were used in the first place to define what is variable in (*ie*, the kinematics of) the theory, and not their  $P$ -mirror or  $T$ -reverse<sup>251</sup> images  $\mathcal{A}^-$ <sup>252</sup>.

At this point we would like to ‘justify’ our re-establishing the aforementioned Principle of Sufficient Reason in a quantum gravity perceived as the dynamics of a non-commutative local quantum causal topology  $\vec{\rho}$  in the kinematical quantum topos  $\text{Qauset}^{\vec{\mathfrak{P}}^{opp}}$ . This ‘justification’ is based on a quantum causal version of the inverse limit mechanism suggested in (Sorkin, 1991, Raptis and Zapatrin, 2000, Mallios and Raptis, 2000) for recovering the locally Euclidean  $C^0$ -manifold topology of the classical spacetime continuum on which GR essentially rests from the Alexandrov-Sorkin poset category or ‘inverse net’  $\vec{\mathfrak{P}}$  of finitary topological substrata, or from  $\vec{\mathfrak{P}}$ ’s contravariant Rota-Zapatrin incidence algebra poset category or ‘direct net’  $\mathfrak{R}$  of quantum topological substrata. In  $\mathfrak{R}$  this mechanism was interpreted as Bohr’s correspondence limit (principle) (Raptis and Zapatrin, 2000) and it was contended that the classical  $C^0$ -spacetime manifold of macroscopic experience together with the commutative algebras  $\Omega^0$  of its local event-determinations arise from some sort of decoherence of the fundamentally a-local quantum topological substrata  $\mathfrak{R}(\vec{\mathfrak{P}}) = \{\Omega_i(P_i)\}$ <sup>253</sup>. Similarly in the

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<sup>250</sup>This author wishes to thank Steve Selesnick for emphasizing all along the importance of this result in numerous private communications (see also below).

<sup>251</sup> $P$  is the parity and  $T$  the time reversal operators of the usual (flat) QFT.

<sup>252</sup>It must be emphasized however that it is quite doubtful whether the  $CPT$  theorem of the usual (flat) QFT on a classical Minkowskian spacetime manifold still holds in the more primordial realm of quantum gravity where spacetime itself, apart from the fact that it is expected to be ‘curved’ in some sense, is also held to be itself a quantum system. Thus one should be careful not to rationalize *a priori* about the initial quantum arrow of time in such  $CPT$  terms which may be valid theoretical notions only for the flat matter quanta of QFT and not for the curved spacetime quanta of quantum gravity. A similar caution is given in (Finkelstein, 1988).

<sup>253</sup>Note that by the association  $\mathfrak{R}(\vec{\mathfrak{P}})$  above, one may simply understand the contravariant functor

quantum causal context, we may assume that the curved directed non-commutative quantum causal topologies in  $\text{Qauset}^{\vec{\mathfrak{P}}^{opp}}$ , at the ideal and non-pragmatic<sup>254</sup> limit of infinite localization of spacetime events<sup>255</sup> in the  $\overrightarrow{\text{Spec}}$  presheaf objects of this quantum topos, yield the classical time-undirected (spatial) topological spacetime manifold  $M$  and the commutative algebra of coordinates of its events. Actually, it would be desirable if one could in fact show that the correspondence limit topos arising from object-wise decohering the quantum topos  $\text{Qauset}^{\vec{\mathfrak{P}}^{opp}}$  is the classical topos  $\mathbf{Sh}(X)$  of sheaves of sets over the region  $X$  of the curved classical spacetime manifold  $M$  of  $GR$ . That this is indeed so, will be shown in (Raptis, 2000d).

At the corresponding dynamical level, the time-asymmetric quantum gravity on  $\text{Qauset}^{\vec{\mathfrak{P}}^{opp}}$  will yield upon decoherence of the a-local directed and curved qausets the usual time-symmetric Einstein equations on the undirected  $M$ . Thus, ‘justification’ for our kinematical quantum time-asymmetry may be the following compelling analogy based on the kinematical mechanism<sup>256</sup> described above:

As the time-symmetric Einstein equations for classical gravity (*ie*, GR) fundamentally rely on the undirected locally Euclidean  $C^0$ -manifold topological space model for spacetime and the commutative algebra of coordinates of its events, so a time-asymmetric quantum gravity must essentially rely on the kinematical or structural directedness of the non-commutative local quantum causality modeled after the quantum causal topos  $\text{Qauset}^{\vec{\mathfrak{P}}^{opp}}$ .

This analogy may be diagrammatically represented as follows

$$\begin{array}{ccc}
 \text{t - asymmetric q - causal kinematics} & \xrightarrow[\text{reason}]{\text{sufficient}} & \text{t - asymmetric q - causal dynamics} \\
 \text{correspondence} \downarrow \text{limit} & & \text{correspondence} \downarrow \text{limit} \\
 \text{t - symmetric c - manifold kinematics} & \xrightarrow[\text{reason}]{\text{sufficient}} & \text{t - symmetric c - Einstein gravity}
 \end{array}$$

with the Principle of Sufficient Reason in the horizontal direction reading ‘time-(a)symmetric kinematics implies time-(a)symmetric dynamics’ and with the afore-

(presheaf) that relates these two poset categories.

<sup>254</sup>See (Raptis and Zapatin, 2000, Mallios and Raptis, 2000).

<sup>255</sup>See (Raptis, 2000b, Mallios and Raptis, 2000).

<sup>256</sup>Again, the characterization of the inverse limit above as being ‘kinematical’ is due to Lee Smolin and Chris Isham. See footnote 8 in the introduction.



mentioned ‘kinematical Bohr correspondence limit mechanism’ in the vertical direction<sup>257</sup>. Due to the discussion above, this mechanism may be called ‘kinematical quantum time-asymmetry breaking’<sup>258</sup>.

Also, from this point of view the unnaturalness of GR in assuming a Lorentzian metric-one that distinguishes in its signature a temporal from three spatial directions-that dynamically propagates on a topologically ‘spatial’ spacetime manifold ( $\mathbb{R}^4$ )-one that regards time as another undirected<sup>259</sup> spatial dimension-is also exposed and in a sense remedied.

We conclude the present paper by remarking on the possibility that the aforementioned ‘quantum topos project’, as approached here from  $A$ -schematic localizations, may prove to be a solid platform for the conceptual unification of quantum logic and quantum gravity.

From a purely mathematical point of view, the conceptual development that led to topos theory *per se* may be roughly summarized as follows: local topological considerations gave rise to sheaf theory which was then employed mainly by Grothendieck, who used purely categorical tools such as ‘representable functors’, to study ring-localizations thus effectively define ‘ $R$ -schemes’ and lay the foundations of topos theory proper. It was mainly Lawvere in the late 60s who recognized the significance of certain special categories, such as those that Grothendieck considered in algebraic geometry<sup>260</sup>, for unifying until then apparently remote from each other and very broad fields of mathematics such as (algebraic) geometry and logic.

<sup>257</sup>In the diagram above, the prefix ‘t’ stands for ‘time’, ‘q’ for ‘quantum’ and ‘c’ for ‘classical’.

<sup>258</sup>To be distinguished from the usual dynamical symmetry breaking scenarios in the usual QFT of matter, as well as certainly worthwhile to be compared with an analogous ‘coherent condensation scenario’ for the time-asymmetric superconducting quantum causal nets in (Finkelstein, 1988). In connection with the latter, Selesnick (1994, 1998) has noted that this condensation process that ‘decoheres’ the ‘purely quantum’ causal net substratum to the (Minkowski space tangent to the) continuous time-symmetric gravitational spacetime  $M$  of macroscopic experience, involves as a first essential step the formation of coherent states of chronon-antichronon Cooper pairs. It is the latter states that a coarse macroscopic observer perceives as ‘the states of the quantum causal net’ when, in fact, as it was noted earlier, only quantum time-directed (*ie*, left-handed or orthochronous)  $sl(2, \mathbb{C})$ -chronons ‘exist’ in the net, not right-handed or antichronous  $\overline{sl}(2, \mathbb{C})$ -antichronons. It may well be the case that the coherent exponentiation of the micro-local Lie algebra  $sl(2, \mathbb{C})$  of chronons to yield the macro-global spin-group  $SL(2, \mathbb{C})$  and its mirror image  $\overline{SL}(2, \mathbb{C})$  which is indistinguishable from it at the coarse macroscopic level of resolution, is the reason for the ‘doubling of macro-spinors’ (*ie*, elements in both the  $S$  and  $S^*$  spaces) thus also for our coarse perception of a macroscopic space/time ( $P/T$ ) symmetry (Steve Selesnick in private communication).

<sup>259</sup>That is to say, two-way and essentially reversible. See introduction and (Finkelstein, 1988, Mallios and Raptis, 2000).

<sup>260</sup>Which categories he (and Tierney) then called ‘topoi’ or ‘toposes’.

It is a generally accepted fact that topoi, regarded as generalized classical logical universes of ‘continuously variable classical sets’<sup>261</sup>, have surprisingly many ‘purely geometrical’ characteristics. In fact, Lawvere (1975) in his celebrated talk in the 1973 Bristol Logic Colloquium went as far as to literally identify ‘algebraic geometry’ with ‘geometric logic’ in the light of topos theory. Indeed, so remarkable is the interplay of logic and geometry in topos theory, especially when sheaves or their descendant schemes are used in order to provide the motivating conceptual background, that the Mac Lane and Moerdijk (1992) book referred to at the back has all four words (*ie*, ‘sheaves’, ‘geometry’, ‘logic’ and ‘topos theory’) in its very title.

On the other hand, it is really an amazing coincidence (in time) that around the same time that topos theory was feverously on the make, Finkelstein (1969, 1979) insisted from a purely physical point of view that as Einstein showed us ‘the physicality and dynamical variability of the geometry of the world’, it was high-time for us to investigate more deeply ‘the physicality and dynamical variability of the logic the world’, as well as “*the possibility that most of the phenomena that we see at higher levels are logical in origin*”<sup>262</sup> (Finkelstein, 1969).

A subsequent series of works of Finkelstein and collaborators culminated in unifying the two fundamental principles of quantum mechanics (uncertainty) and relativity (causality) on quantum logico-algebraic grounds<sup>263</sup>. Quantum Relativity in particular (Finkelstein, 1996), intriguingly contends that “*logics come from dynamics*”.

A quantum topos, like our quantum causal Qauset $\tilde{\mathfrak{P}}^{opp}$ , when compared to its quantum logical analogue  $\text{Set}^{W^{opp}}$  of Butterfield and Isham (1997, 1998), exemplifies precisely the close conceptual interplay between quantum logic and (the time-asymmetric kinematics of) quantum gravity<sup>264</sup>, thus it vindicates Finkelstein’s (and Lawvere’s) vision in physics (and mathematics) that physical (mathematical) logic and physical (mathematical) geometry are not that different afterall. In effect, in the quantum causal topos Qauset $\tilde{\mathfrak{P}}^{opp}$ , especially when its close affinities with the quantum logical topos  $\text{Set}^{W^{opp}}$  of Butterfield and Isham are highlighted in an approach like ours via sheaf and scheme theory, we get the first significant hints about how the

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<sup>261</sup>Like the  $\mathbf{Sh}(X)$  and  $\text{Set}^{W^{opp}}$  ones that we saw in the previous section.

<sup>262</sup>It being understood that the curvaceous geometry of the classical spacetime of macroscopic experience and GR could be ‘explained’ by appealing to a dynamically variable quantum physical logic.

<sup>263</sup>That is, the development of the Grassmann-algebraic Quantum Set Theory and its dynamical descendant Quantum Causal Net Theory, as well as the subsequent integration of these two theories into a more comprehensive Quantum Relativity Theory (Finkelstein, 1996).

<sup>264</sup>Again, perceived as the dynamical theory of a local non-commutative quantum causal topology  $\vec{\rho}$ .

warped quantum logic of the world may be ‘determined’ by the dynamics of a finitary quantum causality, that is to say, by a locally finite, causal and quantal Lorentzian gravity. Qauset <sup>$\mathfrak{P}^{opp}$</sup>  is not that far from being a sound mathematical model of Finkelstein’s deep insight mentioned above that “*logics come from dynamics*” (Finkelstein, 1996). However, a more thorough examination of this deep connection between quantum logic and quantum gravity is left for a paper currently in preparation (Raptis, 2000*d*).

In closing we must stress that the quantum topos project is yet another instance of the general tendency in current theoretical physics of looking at the problem of the quantum structure and dynamics of spacetime from an entirely algebraic point of view. We feel strongly that the ‘innately algebraic’ language of sheaf and scheme theory, as well as of their categorical outgrowth, topos theory, is well suited to implement such an ‘algebraization of quantum gravity’ (Crane, 1995, Isham, 1997, Raptis, 1998, Butterfield and Isham, 2000), thus once again vindicate the prophetic words of Einstein (1956)<sup>265</sup>:

One can give good reasons why reality cannot at all be represented by a continuous field. From the quantum phenomena it appears to follow with certainty that a finite system of finite energy can be completely described by a finite set of numbers (quantum numbers). This does not seem to be in accordance with a continuum theory, and must lead to an attempt to find a purely algebraic theory for the description of reality. But nobody knows how to obtain the basis of such a theory.

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<sup>265</sup>The quotation below can also be found in (Mallios and Raptis, 2000). The reader may also refer to (Crane, 1995) for more quotations of Einstein about ‘an entirely algebraic description of physical reality’.

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<u>C*-quantales</u>	<u><math>\vec{\Omega}</math>-schemes</u>
<b>The algebras</b>	
infinite dimensional non-abelian $C^*$ -algebras $A$ closed under $*$	finite dimensional non-abelian algebras $\vec{\Omega}$ with $\dagger$ -conjugates $\vec{\Omega}^\dagger$
<b>The relevant representations</b>	
equivalence classes of $*$ -irreps	equivalence classes of irreps
<b>The relevant spectra</b>	
maximal: $\text{Max } A$	primitive: $\text{Spec } \vec{\Omega}(\vec{P})$
<b>The quantum points</b>	
closed linear subspaces of $A$ or closed two-sided ideals in $A$	kernels of irreps of $\vec{\Omega}$ or primitive ideals in $\vec{\Omega}$
<b>The non-commutative topology</b>	
based on the non-commutative ‘&’ operation on quantum points	based on the non-commutative ‘o’ operation on quantum points
<b>The underlying logic and geometry</b>	
quantum and not localized/ungauged, hence flat	quantum and localized/gauged, hence curved
<b>The physical interpretation</b>	
undirected quantal spatial topologies between quantum space points	directed quantum causal topologies between quantum spacetime events

Table 1: Comparison between  $C^*$ -quantales and primitive  $\vec{\Omega}$ -finschemes

<u>The neo-classical topos <math>\mathbf{Set}^{W^{opp}}</math></u>	<u>The quantum topos <math>\mathbf{Qauset}^{\vec{\mathfrak{P}}^{opp}}</math></u>
<b>The objects</b>	
‘spatial’ presheaves <b>Spec</b>	‘causal’ presheaves $\overrightarrow{\mathbf{Spec}}$
<b>The subobject classifiers</b>	
a complete distributive lattice (Heyting algebra or locale) $\Omega$	a reticular causal and quantal orthochronous Lorentz-spin algebra $\mathfrak{g}_i = sl(2, \mathbb{C})_i$
<b>The arrows or morphisms</b>	
<b>Spec</b> -presheaf morphisms	$\overrightarrow{\mathbf{Spec}}$ -presheaf morphisms
<b>The relevant quantum points</b>	
Stone spaces $\mathbf{Spec} A$ of Boolean subalgebras $A$ of a quantum lattice $\mathcal{L}(\mathcal{H})$ in the base poset category $W$ of <b>Spec</b>	primitive spectra $\mathbf{Spec} \vec{\Omega}_i(\vec{P}_i)$ in the base poset category $\vec{\mathfrak{P}}$ of $\overrightarrow{\mathbf{Spec}}$
<b>The relevant localizations</b>	
local sections of <b>Spec</b> yield $\mathbf{Spec} A \rightarrow \Omega(A)$ localizations of (truth in) <b>Set</b> over $W$	local sections of $\overrightarrow{\mathbf{Spec}}$ yield $\mathcal{A}_i : \vec{\Omega}_i^1 \rightarrow \mathfrak{g}_i$ gauging of (symmetries of) <b>Qauset</b> over $\vec{\mathfrak{P}}$
<b>The internal logic and topology; twisted geometry</b>	
neo-classical and locally localic, and spatial (non-relativistic); localized, hence warped, thus intransitive local intuitionistic implication $\overset{*}{\Rightarrow}$	locally quantalic ( <i>ie</i> , non-distributive logical and non-commutative topological), and causal (relativistic); gauged, hence curved, thus intransitive local quantum causality $\vec{\rho}$
<b>The physical interpretation</b>	
contextualized constructivistic valuations for variable sets or ‘variable many-valued intuitionistic truth’	kinematics for dynamically variable finitary quantum causal topologies or ‘gravitational quantum causality’

Table 2: Comparison between  $\mathbf{Set}^{W^{opp}}$  and  $\mathbf{Qauset}^{\vec{\mathfrak{P}}^{opp}}$